29/NOV/2007/THU

John Baez: The weak quotient groupoid (cont.)

1. (a) Recall, given a G-set S, our construction of the groupoid

$S//G$: objects are $\{s \in S\}$
morphisms are $\varphi_{g,s}: s \mapsto gs$, where $g.s = g$

Then $\varphi_{g',gs}\varphi_{g,s} = \varphi_{g'g,s}$, $\varphi_{g^{-1},gs} = (\varphi_{g,s})^{-1}$

(b) Then we ~~have~~ want to also ~~prove~~/ verify equivariance:

Say $\varphi: G \to G'$, $\phi: S \to S'$, and the actions are compatible

i.e. $\phi(gs) = \varphi(g)\phi(s)$

i.e.
$$\begin{tikzcd} G\times S \arrow[r, "\cdot"] \arrow[d, "\varphi\times\phi"'] \arrow[dr, phantom, "G"] & S \arrow[d, "\phi"] \\ G'\times S' \arrow[r, "\cdot"] & S' \end{tikzcd}$$

then this is the same as a functor $\Phi: S//G \to S'//G'$, i.e.

$$\begin{tikzcd} s \arrow[r, "(g,s)"] \arrow[d, "\Phi"] & gs \\ \phi(s) \arrow[r, mapsto, "(\varphi(g),\phi(s))"'] & \phi(gs) \end{tikzcd}$$

(c) Why is it called a weak quotient? Well, the (strong) quotient is S/G: $= \{[s]\}$, where $[gs] = [s]$ $\forall s \in S, g \in G$.
i.e. just the set of orbits.

$S//G$ is a weaker version because we now do NOT make isomorphic objects equal!

② ⓐ So now our focus shifts ~~to~~ groupoids, and spans between ~~the~~ them.

We work with a Category-enriched category

= bicategory = weak 2-category

where • objects are groupoids X

• morphisms are spans of groupoids
$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

• 2-morphisms are equivalences of spans (ie equivalence of groupoids) commuting upto natural isomorphism
$$\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[dr] & S' \arrow[dl] \arrow[d] \\ X & Y \end{tikzcd}$$

(Note: A span in any category $\mathcal{C}$ is a diagram $\begin{tikzcd}[cramped] & S \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$ in $\mathcal{C}$)

ⓑ Actually, we're going to be doing ~~...~~ the following decategorification:

Degroupoidification D takes (Groupoids, Spans, iso's of spans) to (vector spaces, linear operators, equality of).

over any fixed ~~field~~ ground field k.

And how does one do this? This was done last time – using the zeroth homology.

[NOTE: We'll only use finite groupoids, for technical purposes]

ⓒ If X is a groupoid, what is its zeroth homology?

If $\mathcal{C}$ is any category, then $\mathcal{C}_0$ = underlying groupoid

$\underline{\mathcal{C}}$ = {isoclasses of objects in $\mathcal{C}$}

and now, $H_0(X, k) = \mathrm{Free}(\underline{X}) = k[\underline{X}]$

(formal fin k-lin comb's of elts of X)

The zeroth cohomology of X is $H^0(X,k) = (H_0(X,k))^* = k^{\underline{X}}$

$= \text{fns}\{\psi : \underline{X} \to k\}$

(These can have infinite support!)

ⓓ eg ① $X = (\mathrm{FinSet})_0$; $H_0(X,k) \cong k[\underline{X}] \cong k[\mathbb{N}_0] \cong k[z]$

and $H^0(X,k) \cong k^{\underline{X}} \cong k^{\mathbb{N}_0} \cong k[[z]]$

② What about $X = (\mathrm{FinVect}_F)_0$? ANS: The same!

Note also that

$$\begin{tikzcd} \mathrm{Fin\,Set} \arrow[rr, "\mathrm{Free}(-)"] \arrow[dr, "1:1"'] & {} \arrow[d, phantom, "\supseteq"] & \mathrm{Fin\,Vect}_F \arrow[dl, "\dim"] \\ & \underline{\mathrm{FinSet}} = \mathbb{N}_0 & \end{tikzcd}$$

commutes.

(3) Now we want to start with a span of finite groupoids

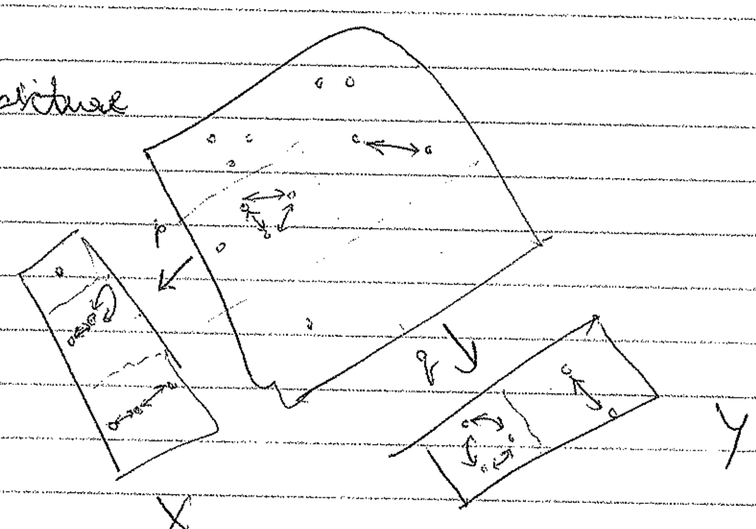
(a)

$$\begin{tikzcd} & S \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

and get a linear operator: ~~Hom~~ $H_0(X) \to H_0(Y)$

ASSUMPTION We'll need to assume $\boxed{\mathrm{Char}(k) = 0}$!

Now, the picture



gives an $\underline{X \times Y}$ - sized matrix of ~~(what sort of) numbers~~ groupoids. This becomes an $\underline{X \times Y}$ - sized matrices of (what kind of) numbers → of fractions.

($\underline{\text{So}}$ k ~~must have~~ ~~co~~ is taken to have characteristic zero.)

This is done by turning each finite groupoid (in our matrix) into a number - its cardinality !

Let's do an example: $X = \bullet\ \bullet\ \bullet$ has $|X|_0 = 3$.

$X = \overset{id}{\bullet} \ \overset{id}{\bullet} \ \overset{id}{\bullet}$? Well $|X/(\mathbb{Z}/2)| = 2$, but

What about $\left| X /\!/ (\mathbb{Z}/2) \right|_0$? Here $\overset{\text{identify}}{\overbrace{\underset{\mathrm{id}}{\circ} \quad \underset{\mathrm{id}}{\circ}}} \longrightarrow \underset{\mathrm{id}}{\circ}$!

So, $\text{Skeleton}\left(X \right) = \underline{X} = \left(\circlearrowleft\!\cdot\!\circlearrowright \quad \cdot\!\circlearrowright \right)$

and hence $\underbrace{\left| X /\!/ (\mathbb{Z}/2\mathbb{Z}) \right|_0}_{\text{groupoid card}} = \underbrace{\left| \circlearrowleft\!\cdot\!\circlearrowright \right|_0}_{\text{Should be } \left(\frac{1}{2}\right)} + \underbrace{\left| \cdot\!\circlearrowright \right|_0}_{=1}$

$= 3/2$!

This is typical of the general picture, since skeleta have only one point components.

~~So, Defn~~

So, define, for a groupoid X, its cardinality $|X|_0$, to be

$$\boxed{|X|_0 := \sum_{[x] \in \underline{X}} \frac{1}{|\mathrm{Aut}_X(x)|}}$$

$\boxed{\text{Exercise}}$ For a finite set S & a fin gp G acting on it,

$$|S /\!/ G|_0 = |S| \,/\, |G| \quad !$$

$\boxed{\text{Proof}}$ Reduce to when G acts transitively. Then $\underline{(S/\!/G)}$ has 1 pt. and $|G|/|S|$ = size of its aut gp.

(Proved)