1/DEC/'20

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(1) (a) Last time JB talked of the cardinality of a tame groupoids → ℝ where the sum

$$|X|_0 := \sum_{[x] \in \underline{X}} \frac{1}{|Aut(x)|} \text{ "converges"?}$$

But we'll / we may also encounter examples where $|X|_0$ may be −ve, & may diverge, & may even be $\in \mathbb{R} \setminus \mathbb{Q}$!

eg: **Exercise** Find $|FINSET|_0$

Ans $\sum_{n=0}^{\infty} \frac{1}{n!} = e$!!

How about $|SET|_0$? Still e? $e+\varepsilon$?

(b) This is now going to be called $|-|_0$: TAME GPDS → ℝ.
"
groupoid cardinality

[Tame groupoids (JB) := groupoids with all $|Aut(x)| < \infty$ and countable # of isoclasses & sum converges]

(c) Next example of this cardinality:

[Set of k-colorings of n balls, and colouring-preserving bijections] → ??

Well, isoclasses / components are precisely

$n_1, \dots, n_k$ balls of colours $1, 2, \dots, k$ resp.

Then groupoid cardinality $= \displaystyle\sum_{n_1, \dots, n_k \geq 0} \frac{1}{n_1! n_2! \cdots n_k!} = \boxed{e^k}$

(d) This is an example of <u>decategorification</u>; its accompanying categorification is not well-understood. eg

Open Problem Find a "nice groupoid" X such that $|X|_0 = \pi$.

(e) **Exercise** Say $X = \{ A \xrightarrow{f} B \text{ of fin sets} \}$, with morphisms being commuting squares

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & C \\ f\downarrow & \circlearrowleft & \downarrow g \\ B & \xrightarrow[\psi]{} & D \end{array}$$

with φ, ψ bij^{n}s.

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(2) The above is NOT the decategorification that we want to use! It's more of the "piggybacking process" that

(a) JD said in his last lecture.

We'll use $\left[\underline{\text{Tame Grpds}} \xrightarrow{|\cdot|_0} \mathbb{R} \right]$ as a morphism-level phenomenon. So, we're going to use

$\underline{\text{Tame Groupoids}} \xrightarrow{\text{de-groupoidification}} \mathbb{C}\text{-VECT}$

$\underline{\text{Tame Span}}$ $\qquad$ $\underline{\qquad}$ LIN OP.

Where a span $G \leftarrow S \rightarrow H$ $=$ $S \downarrow G \times H$ is <u>tame</u> iff ..., eg S, G, H tame $\Rightarrow$ ✓

given by $X \mapsto H_0(X) = \text{Free}_{\mathbb{C}}(\underline{X})$ ← = isoclasses of objects

(b) Actually, to insist on "safety", we'll work only with finite groupoids — in which case we can even work with $\mathbb{Q}_{\geq 0}$ as far as taking the groupoid cardinality goes.

And note that we did not use the groupoid cardinality on the object level, (as we had said).

Rather, given a span $G \xleftarrow{j} S \xrightarrow{k} G'$, we want to get a morphism / lin operator

$$H_0(G) \xrightarrow{j^! \;\to\; H_0(S) \;\to\; k_*} H_0(G'),$$ which we'll define presently.

(c) $G \leftarrow S \rightarrow G'$, is also written as $G' \leftarrow S \rightarrow G$, so we should also get some map $: H_0(G') \longrightarrow H_0(G)$.

And indeed, it's the matrix adjoint = matrix transpose (since all entries are in $\mathbb{R}$) of "$H_0(S)$"! ($(k_* j^!)^T$)

(d) These are the <u>transfer maps</u>; we'll define them now.

$j^! : H_0(G) \to H_0(S)$ sends an (isoclass / component) γ in G, to

$$\sum_{[X]: j([X])=[Y]} \left(\frac{|X|_0}{|Y|_0}\right) X \text{ , where } |X|_0 = \frac{1}{|Aut(X)|} \text{ etc.}$$

where X stands for the basis vector in $H_0(S)$.

eg. j = identity functor ⇒ ~~(~~we need $|Y|_0$ in the denom. (if we had $|X|_0$ above!)

(e) One thing that goes through, is that all the properties of transfers, now go through for our construction.

In particular, the composition of spans leads to composing the relevant linear operators.

(f) We will now ~~(later?) end up~~ correctly stating the theorem that we have been trying to state for some time.

Theorem If X, Y are finite G-sets for a finite group G, then $Free(X \times Y // G) \cong Hom_G(Free(X), Free(Y))$

Pf. Both sides are $\cong$ to ~~the space of Hecke Operators~~

$Free(Orbits_G(X \times Y))$. (Proved)

(g) Now we use spans to get compositions of morphisms etc.

$Hom_G(Free(X), Free(Y)) \otimes Hom_G(Free(Y), Free(Z))$
$\longrightarrow Hom_G(Free(X), Free(Z))$

This would really mean that we've built our category of Perm Reps / Lin Op's out of our Tame gpds / Tame Spans

(4) So, how?

Hom(−) ⊗ Hom(−) should come from

(−) × (−)

So, what span gives $(X \times Y)//G \times (Y \times Z)//G$

$\Uparrow$

??

$\downarrow$

$(X \times Z)//G$?

Equivalently, what gives

$$(X \times Y)//G \leftarrow (?) \rightarrow (Y \times Z)//G$$
$$(?) \downarrow (X \times Z)//G$$

?

Ans Believe it or not, (?) $= (X \times Y \times Z)//G$!!

(Can be verified).

3') **Remark** [Bump, Lie Gps, pg 400] has Four Levels of Math Study, i.e. S_n, $GL_n(\mathbb{F}_q)$, GL_n(local field), GL_n(adele ring of a global field).

We'll try to apply our ideas to these situations