

1/DEC/20

Jim Dolan

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Last time JB talked of the cardinality of a tame groupoids \rightarrow to what the sum

$$|X|_0 = \sum_{[x] \in X} \frac{1}{|\text{Aut}(x)|} \quad \text{"Converges?"}$$

But we'll / we may also encounter examples where

$|X|_0$ may be $-ve$, & may diverge, or may even be $\in \mathbb{R} \setminus \mathbb{Q}$!

eg. Exercise Find $|FINSET|_0$
Ans $\sum_{n=0}^{\infty} \frac{1}{n!} = e$!!

How about $|SET|_0$? Still e ? $e + \epsilon$?

② This is now going to be called $|-|_0 : \text{TAME GPDS} \rightarrow \mathbb{R}$.
" groupoid cardinality.

Tame groupoids (JB) := groupoids with all $|\text{Aut}(x)| < \infty$ and countable # of isobases & sum converges

③ Next example of this cardinality:

Set of k-colourings of n balls, and colour-preserving bijections \rightarrow ??

Well, ~~by~~ isobases / components are precisely

n_1, \dots, n_k balls of colours $1, 2, \dots, k$ resp.

Then groupoid cardinality = $\sum_{n_1, \dots, n_k \geq 0} \frac{1}{n_1! n_2! \dots n_k!} = \boxed{e^k}$

(d) This is an example of declassification; its accompanying categorification is not well-understood. eg

Open Problem Find a "nice groupoid" X such that $|X|_0 = \pi$.

(e) Exercise Say $X = \{ A \xrightarrow{f} B \}$ of functors, with morphisms being commuting squares

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & C \\ f \downarrow & \square & \downarrow g \\ B & \xrightarrow{\psi} & D \end{array} \quad \text{with } \varphi, \psi \text{ bij's.}$$

(2) The above is NOT the declassification that we want to use. It's more of the "piggybacking process" that JD said in his last lecture.

(a) We'll use $[\text{Tame Grpds} \xrightarrow{1:1_0} \mathbb{R}]$ as ~~the~~ a morphism-level phenomenon. So, we'd be going to use

$$\begin{array}{ccc} \text{Tame Groupoids} & \xrightarrow{\text{de-groupoidification}} & \mathbb{C}\text{-VECT} \\ & \xleftarrow{\text{Tame Span}} & \underline{\text{LIN OP.}} \end{array}$$

where a span $\begin{array}{ccc} & S & \\ & \swarrow & \searrow \\ G & & H \end{array} = \begin{array}{ccc} & S & \\ & \downarrow & \\ G \times H & & \end{array}$ is tame iff ...; eg S, G, H tame $\Rightarrow \checkmark$

given by $X \mapsto H_0(X) = \text{Free}_{\mathbb{Q}}(\underline{X})$ = isoclasses of objects.

(b) Actually, to insist on "safety", we'll work only with finite groupoids — in which case we can even work with $\mathbb{Q} \geq 0$ as far as taking the groupoid cardinality goes.

And note that we did not use the groupoid cardinality on the object level (as we had said).

Rather, given a span $S \begin{matrix} \swarrow j \\ G \\ \searrow k \\ G' \end{matrix}$, we want to get a morphism / lin operator

$$H_0(G) \xrightarrow{j^!} H_0(S) \xrightarrow{k_*} H_0(G'), \text{ which we'll define presently.}$$

(c) $S \begin{matrix} \swarrow j \\ G \\ \searrow k \\ G' \end{matrix}$, is also written as $S \begin{matrix} \swarrow j \\ G \\ \searrow k \\ G' \end{matrix}$, so we should also get some map $H_0(G') \rightarrow H_0(G)$.

And indeed, it's the matrix adjoint = matrix transpose (since all entries are in \mathbb{R}) of $H_0(S)$!
 $(j^!)^T = (k_* j^!)^T$

(d) These are the transfer maps; we'll define them now.

$j^! : H_0(G) \rightarrow H_0(S)$ sends an (isoclass / component) γ in G , to

$$\sum_{[X]: j([X])=[Y]} \left(\frac{|X|_0}{|Y|_0} \right) X, \text{ where } |X|_0 = \frac{1}{|\text{Aut}(X)|} \text{ etc.}$$

where X stands for the basis vector in $H_0(S)$.

eg. $f = \text{identity functor} \Rightarrow$ we need $|Y|_0$ in the denom. (if we had $|X|_0$ above!)

(e) One thing that goes through, is that all the properties of transfers, now go through for our construction.

In particular, the composition of Spens leads to composing the relevant linear operators.

(f) We will now ~~finish~~ end up ~~by~~ correctly stating the theorem that we have been trying to state for some time.

Theorem If X, Y are finite G -sets for a finite group G , then $\text{Free}(X \times Y // G) \cong \text{Hom}_G(\text{Free}(X), \text{Free}(Y))$

(PF) Both sides are \cong to ~~the space of G -linear operators~~
 $\cong \text{Free}(\text{Orbits}_G(X \times Y))$. (Proved.)

(g) Now we use Spens to get compositions of morphisms etc.

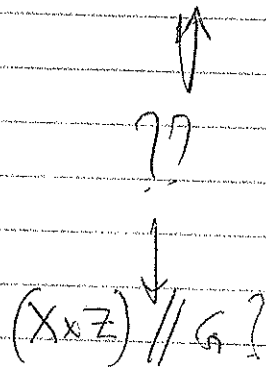
$$\begin{aligned} \text{Hom}_G(\text{Free}(X), \text{Free}(Y)) &\xrightarrow{\otimes} \text{Hom}_G(\text{Free}(Y), \text{Free}(Z)) \\ &\longrightarrow \text{Hom}_G(\text{Free}(X), \text{Free}(Z)) \end{aligned}$$

This would really mean that we've built our category of Perm Reps kind's out of our Tame gpd's Tame Spans

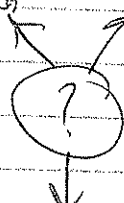
(h) So, how?

$\text{Hom}(-) \otimes \text{Hom}(-)$ should come from
 $\downarrow \quad \swarrow$
 $(-) \times (-)$

So, what span gives $(X \times Y) // G \times (Y \times Z) // G$



Equivalently, what gives $(X \times Y) // G \quad (Y \times Z) // G$



Ans. Believe it or not, $\textcircled{?} = (X \times Y \times Z) // G !!$
 (Can be verified).

3) Remark [Bump, We Gps, § 400] has Four levels of Math Study, i.e.
 $S_n, GL_n(\mathbb{F}_q), GL_n(\text{local field}), GL_n(\text{adele ring of a global field})$.
 We'll try to apply our ideas to these situations.