2/OCT/2007/TUE

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(1) (a) Though we will talk about representation theory — which involves groups and their actions on something — typically linear maps (invertible) on vector spaces —

nevertheless, since we'll deal with geometric rep. theory, we'll see more objects than just vector spaces.

eg. - Diff(M) = group of diffeomorphisms of a smooth manifold
preserve the <u>differentiable structure</u>.

(b) Our guiding principle is:

[<u>Every</u> transformation group is the group of "something-o-morphisms" (!) for some essentially unique "<u>structure</u>".]

(c) <u>What is structure</u>? Very vaguely, it allows us to ~~p~~ tell elements of a set apart. For example an axiomatic theory (involving <u>axioms</u> about <u>abstract predicates</u>) ~~allows us~~ gives structure on some sets, eg. Euclidean geometry.

In some sense, <u>symmetry</u> and structure are dual, & inverse to each other, in the <u>same sense</u> that intermediate fields are dual to Galois (sub) groups: when one of them becomes bigger, the other grows smaller.

So, we ~~these now give~~ see that there are <u>two</u> <u>points</u> of view about talking about structures: one is <u>positive</u> like actually putting an axiomatic system on some set (and predicates), and the other is negative, & recursive/circular/tautological: we just define the ~~symmetric~~ structure as "dual to" (and hence indexed / identified bijectively, via) the group of transformations that is supposed to preserve it.

(2) So, we now seek a positive answer, & an axiomatic theory, to (answer) the following question:

(a) Given a transformation group $G \subseteq S! = \mathrm{Perm}(S)$, i.e.

$$\left[G = \text{group},\ S = \text{set},\ \text{action}: G \hookrightarrow S! \right]$$

find the structure that is preserved by exactly G (from among $S!$).

(b) We'll work towards this goal; ~~but~~ thus, a transformation group ~~generates~~ will give rise to an axiomatic theory.

In fact, the converse is also true, and easier to do!

Given a [Complete (every statement is either true or false) axiomatic theory, with an axiom stating that: the "universe" of the model is bounded by (some) N, $\forall N \in \mathbb{N}$]

~~one can retrieve from it, a~~ we ~~har~~ can conclude that

(1) every statement ~~the~~ is either false or true and provable.

(2) all models of the axiomatic system are isomorphic

(eg. the plane is a model for Euclidean geometry).

Given such an axiomatic system, take some model (which one?!) and look at its automorphism group. Thus, we get a transformation group, and it acts on the universe, and the symmetries that it preserves, are exactly our axiomatic system!

(c) How does one go the other way? I.e. from a finitary transform^n group (i.e. G, S are both finite, where $G \subseteq S!$), find an axiomatic theory!

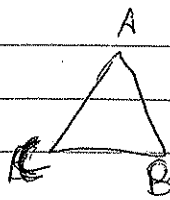
—X—

(3) Back to recovering the ~~symm~~ structure (positively) from the group of symmetries:

We'll introduce the orbi-simplex picture of $G \subseteq S!$:

This is the orbit space of the action of G on the simplex whose vertices are the elements of S.

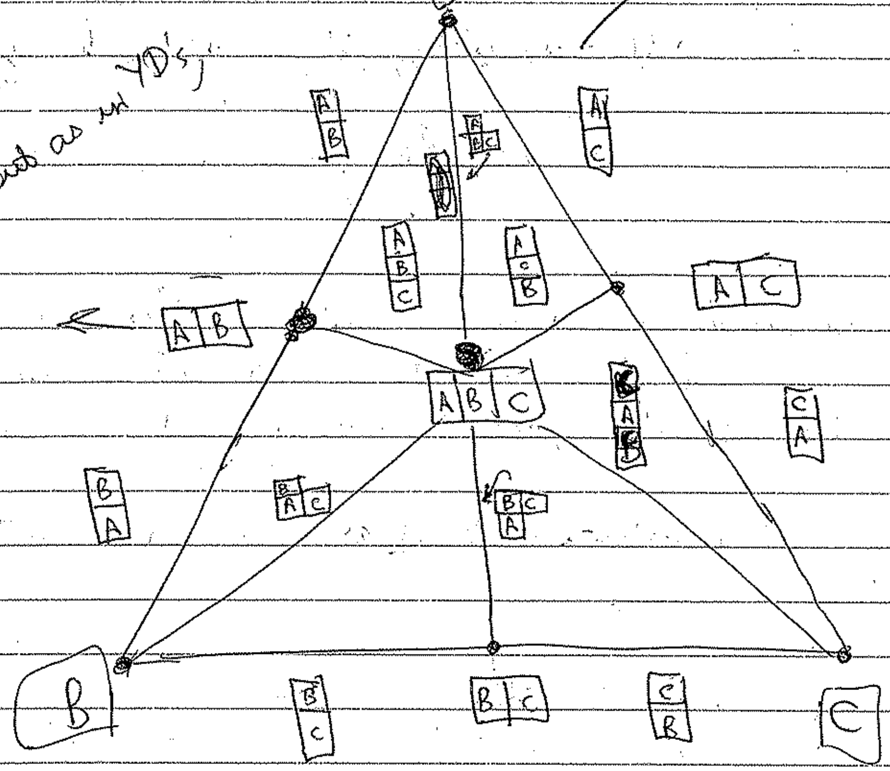
Example: $S = \{A, B, C\}$, G = permutations ~~that~~ fix A. ($\#|G| = 2$).

We draw the simplex 

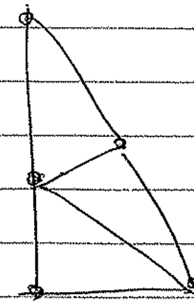
and ~~also the~~ draw the "pre"-picture

Barycentric division

Could be [B|A], but as YD's, we don't order

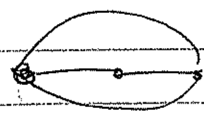


and having labelled the components, we now "fold" along all axes of symmetry obtained from $G \subseteq S_3$ and look at this — this is the orbi-simplex picture S_G, eg for one G here, the picture would be



etc.

and for $\mathbb{Z}_3$,



and for S_3,