

# John Baez : Summary, preview, and more

(1) We've outlined the following result:

**Theorem** There's a category  $\mathcal{C} = [\text{finite groupoids, equivalence classes of spans of finite groupoids}]$  and a functor  $D : \mathcal{C} \rightarrow \text{FinVect}_k$

where  $k$  is any field of characteristic zero ( $\mathbb{C}, \mathbb{R}, \mathbb{Q}, \dots$ ) given by

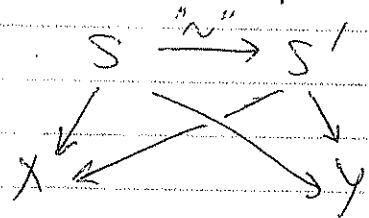
$$X \longmapsto H_0(X) = \text{Free}_k(\underline{X})$$

$$\left( \begin{array}{ccc} & S & \\ p \swarrow & & \searrow q \\ X & & Y \end{array} \right) \longmapsto H_0(X) \xrightarrow[p]{p'} H_0(S) \xrightarrow[q]{q'} H_0(Y)$$

transfer

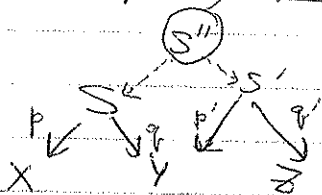
(2) There are subtleties involved in all this! Here's some of them:

(i) Two spans  $X \rightarrow Y$  are equivalent if  $\exists$  equivalence of groupoids making the diagram commute upto natural isom.



(ii) One needs to check that  $D$  is well-defined on morphisms/spans! (i.e. - equivalence spans give equal operators).

(iii) Composing spans: We compose spans of groupoids via weak pullbacks. Thus,  $S' \circ S = \text{groupoid } S''$  which we now describe:



An object of  $S''$  is  $\bullet$  an object  $s$  of  $S$   
 $\bullet$  an object  $s'$  of  $S'$   
 $\bullet$  an isom.  $\alpha: q(s) \rightarrow p'(s')$

And a morphism should take  $q(s_1) \xrightarrow[\sim]{\alpha_1} p'(s'_1)$

↓

$q(s_2) \xrightarrow[\sim]{\alpha_2} p'(s'_2)$

So, the "correct" guess is: start with isomorphisms  $[ f : S_1 \rightarrow S_2 \quad \& \quad f' : S'_1 \rightarrow S'_2 ]$ , and apply the functors  $[ q \quad \& \quad p ]$  resp.; so that

$$\begin{array}{ccc}
 q(s_1) & \xrightarrow[\sim]{\alpha_1} & p'(s'_1) \\
 q(f) \downarrow ? & & p'(f') \downarrow ? \\
 q(s_2) & \xrightarrow[\sim]{\alpha_2} & p'(s'_2)
 \end{array}
 \quad \text{Commutative.}$$

Remarks ① This composition is associative & unital up to equivalence

② This is the "low-brow" approach; the high-brow approach uses  $\mathcal{W}$  weak 2-categories.

② All this was supposed to be a pre-requisite to help talk about Hecke operators! So we now address that question.

① Using  $D : \mathcal{C} \rightarrow \text{FinVect}_k$ , we can turn a category  $\mathcal{A}$  (enriched over  $\mathcal{C}$ ), into a category  $\overline{D}(\mathcal{A})$  (enriched over  $\text{FinVect}_k$ ).  
 Roughly speaking,  $\overline{D}(\mathcal{A})$  has the same objects as  $\mathcal{A}$ , and

$$\underbrace{\text{hom}_{\overline{D}(\mathcal{A})}(x,y)}_{\text{is now in FinVect}_k} \quad \parallel \quad \underbrace{D(\text{hom}_{\mathcal{A}}(x,y))}_{\text{is in } \mathcal{C}}$$

(b) Let us re-specify what enriching is, via a specific example.

(i)  $\mathcal{C} = \text{monoidal category } (\otimes)$  | eg  $\mathcal{C} = [\text{finite groupoids, eq. classes of spans}]$   
 $\otimes = \text{Cartesian prod of groupoids}$   
 $= \text{usual product of categories}$

(ii)  $A_{\mathcal{C}} = A$  enriched over  $\mathcal{C}$ , has

- a class of objects ( $= \text{Ob}(A)$ )  $\equiv$  the same here
- for any  $x, y$  objects, ~~finite groupoid~~ an object  $\text{hom}_A(x, y)$  of  $\mathcal{C} \equiv$  a finite groupoid  $\text{hom}(x, y)$

- for any  $x, y, z$  objects,  
 $\circ : \text{hom}_A(x, y) \otimes \text{hom}_A(y, z) \rightarrow \text{hom}_A(x, z)$ ,  $\equiv$  replace  $\otimes$  by  $\times$ .  
 a morphism in  $\mathcal{C}$

- for any object  $x$  in  $A$ ,  
 $\text{id}_x : \mathbb{1} \rightarrow \text{hom}_A(x, x)$   $\equiv$  ---  
 $\parallel$   $\searrow$   
 terminal object  $\rightarrow$  morphism in  $\mathcal{C}$

- associativity, left/right unit laws  $\equiv$  ---

(c) We'll actually do a key example now, of  
 Pick a finite group  $G$ . Then we can attach to  
 this, the category Hecke( $G$ ), enriched over  $\mathcal{C} = (\text{fin gpds}, \dots)$

↳ Objects of Hecke( $G$ ) are finite  $G$ -sets  $X, Y, \dots$   
 ↳ Morphisms are  $\text{hom}_{\text{Hecke}(G)}(X, Y) = (X \times Y) // G$  ~~...~~

So this is a finite groupoid.

↳ Compositions? Unit?

(d) But first we take a detour and ask what this has to do with Hecke operators.

We call  $(X \times Y) // G$  the Hecke groupoid of  $X$  &  $Y$ ; its components are simply the  $G$ -orbits of  $X \times Y$  (this is true for any  $G$ -set:  $S // G = S/G$ ).

$$\text{Hence } \text{hom}_{\overline{\mathbb{D}}(\text{Hecke}(G))}(X, Y) = \mathbb{D}(\text{hom}_{\text{Hecke}(G)}(X, Y)) \\ = H_0(\text{hom}_{\text{Hecke}(G)}(X, Y)) \cong \mathbb{k}^{(X \times Y) / G}$$

Elements of  $(X \times Y) / G$  are also called "atomic invariants relations" between  $X$  &  $Y$ .

and Jim D — proved a long time ago that these ↓ give a basis of the vector space

$$\text{hom}_G(\mathbb{k}^X, \mathbb{k}^Y)$$

(↪ perm. reps of  $G$ )

and this basis is also called

"Hecke operators" !!

(e) **Upside**  $\text{hom}_{\overline{\mathbb{D}}(\text{Hecke}(G))}(X, Y) \cong \text{hom}_G(\mathbb{k}^X, \mathbb{k}^Y) = \mathbb{D}((X \times Y) // G) \\ = \mathbb{k}^{(X \times Y) / G} = H_0((X \times Y) // G)$

and we will now formally state this as a theorem!

(f) **Fundamental Theorem of Hecke Operators**

then  $\overline{\mathbb{D}}(\text{Hecke}(G)) \cong \text{Fin. Perm. Rep.}(G)_{\mathbb{k}}$

$\mathbb{C}$ -enriched ↗

FinVect $_{\mathbb{k}}$ -enriched ↗

If  $G$  is a finite group,  $\mathbb{k}$  = field w/  $\text{char}(\mathbb{k}) = 0$ ,

so that the Upside (d) above, holds.

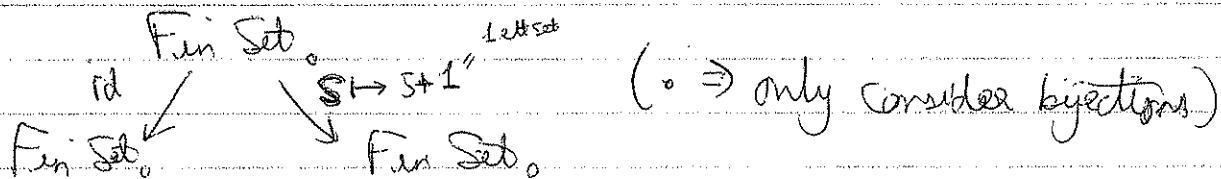
—X— (Turn four pages after this, to finish the notes!) —X—

## Baez (Cont.) (from 4 pages behind)

③ What next? Let's go back to "Pascal's Triangle".

Consider the following span of non-finite groupoids

②



( $\circ \Rightarrow$  only consider bijections)

Though the above theory may not apply (since while turning the span into a morphism, the groupoid cardinality may diverge! But we proceed nevertheless.

$$\text{becomes } \text{Ho}(\text{Fin Set}_0) \xrightarrow{1!} \text{Ho}(\text{Fin Set}_0) \xrightarrow{k_x} \text{Ho}(\text{Fin Set}_0)$$

$$\text{But as seen earlier, } \text{Ho}(\text{Fin Set}_0) = \text{Free}_k(\mathbb{N}_0) = k[x]$$

and  $1! = 1$  because  $1!$  is a contravariant functor etc.

and  $S \mapsto S+1$  is just mult by  $x$ .

④ Upshot So, the above span just becomes the morphism  $\text{mult}_x : k[x] \rightarrow k[x]$ .

Question? What does the "other way" span look like?

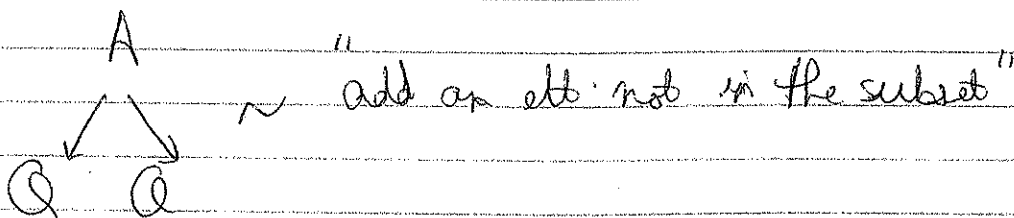
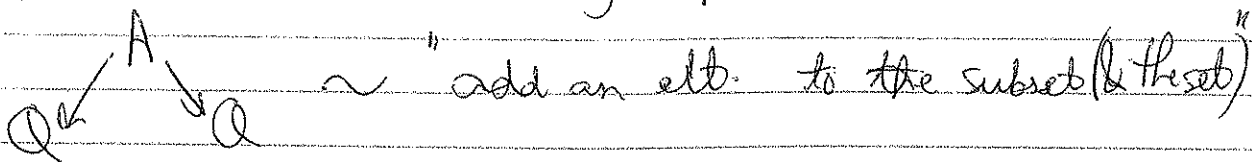
Thinking about transfers, it will turn out that it is  $\frac{d}{dx}$

$\rightarrow$  these are also called the creation & annihilation operators respectively.

Thus, we will be doing some sort of categorified quantum mechanics next quarter.

Next, consider  $\mathcal{Q} = [\text{finite sets equipped with a subset}]_0$   
 $= \coprod_{n \geq k \geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_0$

Here, we have two "interesting" spans:



These give 2 linear operators - which we will denote by "mult by X" & "mult by Y"; thus we need to somehow (maybe slightly non-rigorously!) think of  $\text{Hom}(\mathcal{Q})$  as  $\mathbb{R}\langle X, Y \rangle$

Then we can categorify the binomial theorem & Pascal's  $\Delta$ !

(d) Similarly, for the  $q$ -deformed version  $\text{Fin Set} \mapsto \text{Fin Vect}_{\mathbb{F}_q}$ , we'll categorify the  $q$ -binomial theorem &  $q$ -Pascal  $\Delta$ .

(e) Then go on to the Multinomial Theorem.

(f) These also give quantum planes, quantum  $n$ -space  $\rightarrow$  [being in the action of quantum space] eg  $GL_q(n, k)$ . Then categorify THAT! OR, "upper  $\Delta$ " etc.