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John Baez : Summary, preview, and more

(1) We've outlined the following result:

$\boxed{\text{Theorem}}$ There's a category $\mathcal{C} = [\text{finite groupoids, equivalence classes of spans of finite groupoids}]$

(a) and a functor $D : \mathcal{C} \to \mathrm{FinVect}_k$

where k is any field of characteristic zero ($\mathbb{C}, \mathbb{R}, \mathbb{Q}, \dots$) given by

$$X \longmapsto H_0(X) = \mathrm{Free}_k(\underline{X})$$

$$\left(\begin{tikzcd} & S \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}\right) \longmapsto H_0(X) \xrightarrow[\underset{\text{transfer}}{\shortparallel}]{p^!} H_0(S) \xrightarrow{q_*} H_0(Y)$$

(b) There are subtleties involved in all this! Here's some of them:

(i) Two spans : $X \to Y$ are equivalent if $\exists$ equivalence of groupoids making the diagram commute up to natural isom.

$$\begin{tikzcd} S \arrow[rr, "\sim"] \arrow[d] \arrow[drr] & & S' \arrow[d] \arrow[dll] \\ X & & Y \end{tikzcd}$$

(ii) One needs to check that D is well-defined on morphisms / spans! (i.e. equivalence spans give equal operators).

(iii) Composing spans: We compose spans of groupoids via weak pullbacks. Thus,

$$\begin{tikzcd} & & S'' \arrow[dl, dashed] \arrow[dr, dashed] & & \\ & S \arrow[dl, "p"'] \arrow[dr, "q"] & & S' \arrow[dl, "p'"'] \arrow[dr, "q'"] & \\ X & & Y & & Z \end{tikzcd}$$

$S' \circ S =$ groupoid S'' which we now describe:

An object of S'' is an object s of S
- an object s' of S'
- an isom. $\alpha : q(s) \to p'(s')$

and a morphism should take

$$\begin{array}{ccc} q(S_1) & \xrightarrow[\sim]{\alpha_1} & p'(S_1') \\ \downarrow & & \\ q(S_2) & \xrightarrow[\sim]{\alpha_2} & p'(S_2') \end{array}$$

So, the "correct" guess is: start with isomorphisms $[f : S_1 \to S_2 \;\&\; f' : S_1' \to S_2']$, and apply the functors $[q \;\&\; p]$ resp., so that

$$\begin{array}{ccc} q(S_1) & \xrightarrow[\sim]{\alpha_1} & p'(S_1') \\ {\scriptstyle q(f)}\downarrow\wr & & {\scriptstyle p'(f')}\downarrow\wr \\ q(S_2) & \xrightarrow[\sim]{\alpha_2} & p'(S_2') \end{array}$$

Commutes.

Remarks: ① This composition is associative & unital up to equivalence

② This is the "low-brow" approach; the high-brow approach uses weak 2-categories.

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② All this was supposed to be a pre-requisite to help talk about Hecke operators! So we now address that question.

ⓐ Using $D : \mathcal{C} \to \text{FinVect}_k$, we can turn a category $\mathcal{A}_{\mathcal{C}}$ (enriched over $\mathcal{C}$), into a category $\overline{D}(\mathcal{A})_{FV_k}$ (enriched over FinVect_k).

Roughly speaking, $\overline{D}(\mathcal{A})$ has the same objects as $\mathcal{A}$, and

$$\underbrace{\hom_{\overline{D}(\mathcal{A})}(x,y)}_{\text{is now in FinVect}_k} = D(\underbrace{\hom_{\mathcal{A}}(x,y)}_{\in \text{Ob}(\mathcal{C})})$$

(b) Let us re-specify what enriching is, via a specific example.

(i) $\mathcal{C}$ = monoidal category ($\otimes$) | eg $\mathcal{C}$ = [finite groupoids, eq. classes of spans]
$\otimes$ = Cartesian prod of groupoids
= usual product of categories

(ii) $\mathcal{A}_{\mathcal{C}}$ = $\mathcal{A}$ enriched over $\mathcal{C}$, has

- a class of objects ($= Ob(\mathcal{A})$) = the same here
- for any x, y objects, ~~finite groupoid~~ an object $hom_{\mathcal{A}}(x,y)$ of $\mathcal{C}$ = a finite groupoid $hom_{\mathcal{A}}(x,y)$
- for any x, y, z objects,
  $\circ : hom_{\mathcal{A}}(x,y) \otimes hom_{\mathcal{A}}(y,z) \to hom_{\mathcal{A}}(x,z)$, a morphism in $\mathcal{C}$ = replace $\otimes$ by $\times$.
- for any object x in $\mathcal{A}$,
  $id_x : \underset{\substack{\| \\ \text{terminal object}}}{\mathbb{1}} \longrightarrow hom_{\mathcal{A}}(x,x)$ → morphism in $\mathcal{C}$ = ---
- associativity, left/right unit laws = ---

(c) We'll actually do a key example now, of
Pick a finite group G. Then we can attach to this, ~~the~~ the category
$\underline{Hecke(G)}$, enriched over $\mathcal{C}$ = (fin gpds, ...)

← Objects of Hecke(G) are finite G-sets $X, Y, \ldots$
← Morphisms are $hom_{Hecke(G)}(X,Y) = (X \times Y)//G$ [illegible]
So this is a finite groupoid.
← Compositions? Unit?

(d) But first we take a detour! and ask what this has to do with Hecke Operators. We call $(X \times Y)//G$ the <u>Hecke groupoid</u> of X & Y; its components are simply the G-orbits of $X \times Y$ (this is true for any G-set: $\underline{S//G = S/G}$).

$$\text{Hence } \hom_{\bar{D}(\mathrm{Hecke}(G))}(X,Y) = D(\hom_{\mathrm{Hecke}(G)}(X,Y))$$
$$= H_0(\hom_{\mathrm{Hecke}(G)}(X,Y)) \cong k^{(X \times Y)/G}$$

Elements of $(X \times Y)/G$ are also called "atomic invariant relations" between X & Y.

and Jim D — proved a long time ago that these give a basis of the vector space

$$\hom_G(k^X, k^Y)$$
($\to$ perm reps of G)

and this basis is also called "<u>Hecke Operators</u>" !!

(e) $\boxed{\text{Upshot}}$ $\hom_{\bar{D}(\mathrm{Hecke}(G))}(X,Y) \cong \hom_G(k^X, k^Y) = D(X \times Y//G)$
$= k^{(X \times Y)/G} = H_0(X \times Y//G)$

and we will now formally state this as a Theorem!

(f) $\boxed{\text{Fundamental Theorem of Hecke Operators}}$ If G is a finite group, k = field w/ $\mathrm{char}(k) = 0$, then $\bar{D}(\mathrm{Hecke}(G)) \cong \mathrm{FinPermRep}(G)_k$,

$\mathcal{C}$-enriched, $\mathrm{FinVect}_k$-enriched

So that the Upshot (e) above, holds.

— X — (Turn four pages after this, to finish the notes!)

— X —

Baez (Cont.) (from 4 pages behind)

(3) What next? Let's go back to "Pascal's Triangle":

Consider the following span of non-finite groupoids

(a)

$$\begin{tikzcd} & \text{Fin Set}_0 \arrow[dl, "\text{id}"'] \arrow[dr, "S \mapsto S+1" "\text{1 elt set}"] & \\ \text{Fin Set}_0 & & \text{Fin Set}_0 \end{tikzcd}$$

($\circ \Rightarrow$ only consider bijections)

Though the above theory may not apply (since while turning the span into a morphism the groupoid cardinality may diverge! But we proceed nevertheless.

becomes $H_0(\text{Fin Set}_0) \xrightarrow{1^!} H_0(\text{Fin Set}_0) \xrightarrow{p_*} H_0(\text{Fin Set}_0)$

But as seen earlier, $H_0(\text{Fin Set}_0) = \text{Free}_k(\mathbb{N}_0) = k[x]$

$[n] \mapsto x^n$

and $1^! = 1$ because $1^!$ is a contravariant functor etc.

and $S \mapsto S+1$ is just mult by X.

(b) Upshot: So, the above span just becomes the morphism $\text{mult}_x : k[x] \circlearrowleft$.

Question? What does the "other way" span look like?

Thinking about transfers, it will turn out that it is $\frac{d}{dx}$

→ these are also called the creation & annihilation operators respectively.

Thus, we will be doing some sort of Categorified quantum mechanics next quarter.

(c) Next, consider $Q = [\text{finite sets equipped with a subset}]_0$

$$= \coprod_{n \geq k \geq 0} \left[\begin{matrix} n \\ k \end{matrix}\right]_0$$

Here, we have two "interesting" spans:

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ Q & & Q \end{tikzcd}$$ ~ "add an elt. to the subset (& the set)"

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ Q & & Q \end{tikzcd}$$ ~ "add an elt. not in the subset"

These give 2 linear operators — which we will denote by "mult by X" & "mult by Y"; thus we need to somehow (maybe slightly nontrivially!) think of $H_0(Q)$ as $\mathbb{R}[X, Y]$

Then we can categorify the Binomial Theorem & Pascal's Δ!

(d) Similarly, for the q-deformed version $\text{FinSet} \mapsto \text{FinVect}_{\mathbb{F}_q}$, we'll categorify the q-Binomial Thm & q-Pascal Δ.

(e) Then go on to the Multinomial Theorem.

(f) These also give quantum plane, quantum n-space → [brings in the action of quantum groups, e.g. $GL_q(n,k)$ - Then categorify THAT! OR, "upper Δ^{r}" etc.]