OCT/2007/THU

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$\boxed{\text{Last Time}}$: (G a finite group)

Hecke operators → $\left\{ \begin{array}{c} \text{G-equivariant Linear Operators} \\ \text{Between Permutation Representations} \end{array} \right\}$

$$\|?$$

$\left\{ \begin{array}{c} \text{Free vector space on the G-orbits on} \\ \text{the Cartesian product of G-sets} \end{array} \right\}$

(a) $\boxed{\text{Lemma}}$

ie (?) $\text{Free}(\text{Orbit}(X \times Y)) \cong \text{Hom}_G(\text{Free}(X), \text{Free}(Y))$

where X, Y are G-sets.

(b) We'll try and relate this notion of Hecke Operators (made more precise later), with that of Hecke algebras.

We already said that G-orbits on $X \times Y$ are trying to describe "geometrico-logical" relationships between types of geometrical figures.

Today we give more examples of this notion, notably the ones from the previous lectures (eg flags in various contexts).

(c) $\boxed{\text{Example}}$ G = isometries of cube, preserving its location in space.

Then $|G| = 48$, for:

each vertex goes to any of 8, edges → 3 orientations, 2 orientations. $\Rightarrow 48$

Let's take G-set $X = \{\text{corners}\}$, $|X| = 8$
G-set $Y = \{\text{edges}\}$, $|Y| = 12$

Then the logical possibilities are (C = Corner, E = edge)

(1) C lies on E $\rightarrow 3$
(2) C, E lie on one face (and no better) $\rightarrow 6$
(3) C, E lie on the cube (— do —) $\rightarrow 3$

(d) Now we need to make this mathematically precise.
But before that, let's pick only orientation-preserving isometries $\rightarrow$ so (new G) = 24-elts.

(e) <u>How does the group come in?</u> In the following sense:

(i) If we apply $g \in G$ (any g) to a C & E satisfying (1) & (2) & (3), we end up with some (other) C & E still satisfying the same conditions.

(ii) <u>Any two</u> (C, E) pairs satisfying, say, (2), are in one G-orbit.

(f) The correspondences in (a) can give interesting "translation" of concepts in one setting, to another.

For example, composing operators $\rightarrow$?

Composing two orbits need not give just one orbit

~~Here's~~ eg product of basis elts in an algebra need not give another basis elt, merely a linear combination.

eg. Corner $\xrightarrow{\text{lies on}}$ edge $\xrightarrow{\text{contains}}$ Corner $\not\Rightarrow$ Corner = Corner

That's only one orbit, but there's another orbit possible:

Corners share an edge (but no better):

~~② Next example: What JB has been doing: Grassmannians~~

~~② Proof of Lemma~~

~~RTP $\text{Free}(\text{Orbit}(X\times Y)) \cong \text{Hom}_G(\text{Free}(X), \text{Free}(Y))$~~

~~Pf We'll try to find $\rightleftarrows$ maps, inverse to each other.~~

~~$\rightarrow$ Enough to find on basis elts:~~

~~So say (x,y)~~

② $\boxed{\text{Proof of Lemma}}$ RTP: $\text{Free}(\text{Orbit}(X\times Y)) \cong \text{Hom}_G(\text{Free}(X), \text{Free}(Y))$

ⓐ REDUCTION: $X = \coprod_{i\in I} X_i$, $Y = \coprod_{j\in J} Y_j$

where X_i, Y_j are transitive ~~letters~~ orbits.

Then $\text{Free}(X) = \bigoplus_{i\in I} \text{Free}(X_i)$ as G-reps.

and similarly for $\text{Free}(Y)$

AND $\text{Orbit}(X\times Y) = \coprod_{(i,j)\in I\times J} \text{Orbit}(X_i\times Y_j)$

(b) So $\boxed{\text{ETS}}$ $\text{Free}(\text{Orbit}(X \times Y)) \cong \text{Hom}_G(\text{Free}(X), \text{Free}(Y))$ for ~~transitive~~ X, Y.

$\boxed{\text{Proof}}$ We'll show maps both ways; they're 2-sided inverses to each other.

$\boxed{\rightarrow}$ On a basis: $\{g(x,y) : g \in G\} \longmapsto \left[\tau : [gx] \mapsto [gy] \;\; \forall g \in G\right]$

where $\{[x] \in X\}$ spans $\text{Free}(X)$

$\boxed{\leftarrow}$ $\left(\tau : [x] \mapsto [\tau(x)]\right) \in \text{Hom}_G(\text{Free}(X), \text{Free}(Y))$

(with $\tau(x) \in Y$) forms a basis of $\text{Hom}_G(\text{Free}(X), \text{Free}(Y))$.

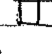
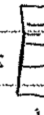
So, we again give the map on this basis:

$[\tau(x) \mapsto y] \longrightarrow \{g(x,y) : g \in G\}$

One can check that these two maps are inverses of one another. Hence the isomorphism is proved.

(Proved)

—— X ——

(3) Next example: $G = GL(4, F)$, $X =$  D1 - flags, $Y =$  D2 - flags

X: $= L' \subseteq FP^3$; Y: $= P \leq L \leq PL \leq FP^3$

The Config's are:

(a) $L' = L$	(c) (i) $P \leq L'$	(d) $L' \cap L \neq \emptyset$
(b) $P \lneq L' \leq PL$	(ii) $L' \leq PL$	(e) L' generic (NOTE: $L' \cap PL \neq \emptyset$)

"Ask JB for explanations of the 2 "Remarks" from his previous lecture (see notes!).