RCT/2007/TUE

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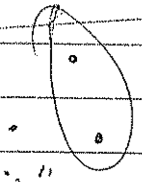
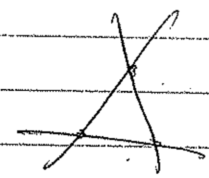
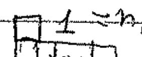
(1) We continue with our "comparison chart":

Set Theory	Projective Geometry
Finite sets	Finite dim vector spaces
n	F^n or FP^{n-1} (1-dim subspaces) (points)
e.g. $n=3$  Here, a "line" is just 2 pts	
points, pairs, triples, ...	points, lines, planes, ...
D-flags	D-flags

(2) We now do our homework:

(a) Say, $D =$  $1 = n_1$. Then

(D-flags on X = a set) = points of X!

$\# |D(n)| = n$

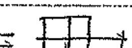
D-flags on F^n = points of FP^{n-1}

Here, $\# = [n]_q$ b/c

$$FP^{n-1} = \frac{F^n - \{0\}}{F - \{0\}} \xrightarrow{\text{decategorify}} \frac{q^n - 1}{q-1}$$

Categorification (much harder)

[So, we're actually replacing #'s (q-ints) by sets!]

(b) $D =$  $n_1 = 2$

$\Rightarrow |D(5)| = \binom{5}{2}$

$= \frac{5!}{2!\,3!}$

It's ~~hard~~ better to look for $\boxed{\text{sets}}$!

that mean $5!$

denominator is

This is often called the ~~Young~~ Young subgroup → these are the perm's precisely preserving our D-flag

(So, S_5 acts transitively on all D-flags, and denom. = stab (pt), so ✓)

We'll now try to count here using the same philosophy!

So, first find a transitive gp action and then find stabilizer ---

The way to go from S_5 to $\mathbb{F}_q$ is $GL(5, F)$, which acts transitively on $D(F^5)$ for sure.

Then it's enough to show/find stabilizer subgp of $\begin{pmatrix} * \\ * \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

But this is $\left(\begin{array}{c|c} GL_2 & gl_{2\times 3} \\ \hline 0 & GL_3 \end{array}\right)$ → which is an example of

Defn A parabolic subgroup of GL_n is one that preserves some D-flag on F^n.

The way now is to count various subgps:

$$|D(F^5)| = \frac{|GL(5,F)|}{|(GL(2,F) \times GL(3,F)) \ltimes F^{2\times 3}|}$$

(A,B) acts on maps: $F^2 \to F^3$ via

$(A,B) \circ \varphi(-) = B\varphi(A-)$ etc.

$\boxed{\text{Remark}}$ The reason we put in $\ltimes$ here, is that eventually we will want to categorify everything, and want both sides to be isom. in some category. We want the "strongest possible $\cong$"

If we use general $\mathbb{F}$ instead of $\mathbb{F}_q$, the $\times$ are going to be (locally trivial) bundles w/ fibers — where we take a bundle every time we make a choice.

<u>So</u> — what's $|GL(n, \mathbb{F}_q)|$?

$\boxed{\text{Ans}}$ Look at it column by column. The first column is nonzero so $(q^n - 1)$

The second is anything not from the line so $q^n - q$

etc.

so $|GL(n, \mathbb{F}_q)| = |(\mathbb{F}_q^n - \{0\}) \times (\mathbb{F}_q^n - \mathbb{F}_q^1) \times \cdots \times (\mathbb{F}_q^n - \mathbb{F}_q^{n-1})|$

$= (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$

$= \cdots = \boxed{[n]!_q \, (q-1)^n \, q^{\binom{n}{2}}}$

So what we now do is

$$|D(\mathbb{F}_q^5)| = \frac{[5]!_q (q-1)^5 q^{\binom{5}{2}}}{[2]!_q (q-1)^2 q^{\binom{2}{2}} \cdot [3]!_q (q-1)^3 q^{\binom{3}{2}} \cdot q^{2 \cdot 3}}$$

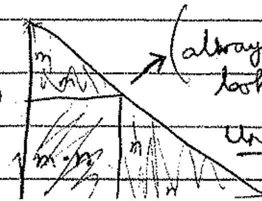
This becomes a product of 3 factors:

(a) <u>the q-binomial coefficient</u> $\dfrac{[5]!_q}{[2]!_q [3]!_q} = \binom{5}{2}_q$

(b) $(q-1)^5$ cancel out

(c) q^{-} <u>also cancel</u>: $\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + (m \cdot n)$

because in pictures,



(always exclude diag, & only look at $\triangle = \frac{1}{2}\square$, b/c <u>unordered</u> <u>non equal</u> pairs)

(3) This happens in general too: For $D =$ 

we have $D(F^n) :=$ Grassmanian of l-dim subspaces of F^n.

$= \mathrm{Gr}_F(n,l)$, which we now call $\boxed{\binom{n}{l}_F}$!

Then one can do this in general: for sets and for vector spaces. (See the analysis in an earlier class.)

So we end up with
$$\binom{n}{n_1, \dots, n_k}_F \quad \& \ (F = \mathbb{F}_q) = \checkmark$$

(4) —X— In connection with above policies:

$\boxed{\text{Defn}}$ For any n-box Uncombed YD D, $D(F^n)$ is denoted by $\binom{n}{n_1, \dots, n_k}_F$

$\boxed{\text{Remarks}}$ (1) Just like $\binom{n}{n_1, \dots, n_k} \in \mathbb{Z}$, and not just $\in \mathbb{Q}$, why are $\binom{n}{n_1, \dots, n_k}_q \in \mathbb{Z}_{\geq 0}[q]$, and not just $\in \mathbb{Q}(q)$?

(2) As $\binom{n}{n_1, \dots, n_k}$ is symmetric in n_i's, is there a bij^n b/w say $\binom{5}{3,3}_q \leftrightarrow \binom{5}{3,2}_q$?

(5) More gleaning of information / observations about $[n]_q$'s:

(a) $[n]_q = 1 + q + q^2 + \cdots$ categorifies ... follows:

Schubert

$\mathbb{F}P^{n-1} = [n]_F$ has a Cell decomposition: $1+F+F^2+\cdots$

using ~~lines~~ pt, line, —

(b) Why does $\boxed{\text{Remark ①}}$ hold? It comes from cohomology of Schubert cells. →

~~Even~~ Say Grassmanian, since every multinomial coeff $= \prod$ binom

So, look at $\binom{n}{l}_q$. Then $Gr_F(n, l) = \coprod$ Schubert cells,

and # Cells n_d of degree d gives

$$\sum n_d q^d = \checkmark$$

Hence $\binom{n}{l}_q \in \mathbb{Z}_{\geq 0}[q]$, not just $\mathbb{Q}(q)$!

—X—

(c) $\boxed{\text{Example}}$ $\binom{4}{2}_q$ = # 2-planes in $\mathbb{F}^4$
$= q^0 + q^1 + 2q^2 + q^3 + q^4$

Or, speaking projectively, # lines in 3-(proj-)space

~~eg~~ Thus, q^0 = most special family
q^1 = slightly less special
q^2 = slightly lesser —

q^4 = most generic, hence $|F^4|$-many!

In fact, $q^0 \rightarrow$ our line is the line

$q^1 \rightarrow$ our line contains pt. & $= \cancel{x}$ axis | and NO BETTER

$q^2 \rightarrow$ our line $\subseteq$ plane | "

$q^2 \rightarrow$ our line contains the point | "

$q^3 \rightarrow$ our line intersects the ~~plane~~ line | " (i.e. not @ the point etc.)

$q^4 \rightarrow [\,b\,]$ and no better!

(d) Remark (2) will be addressed later on in the course...

$\boxed{\text{NOTE}}$ Both Remarks ~~are~~ only need to be explained for q-binomial coefficients, because every multinomial coeff is a product of binomial coeffs.

Moreover, this should transfer over to the picture, world of sets, via some sort of functoriality.