

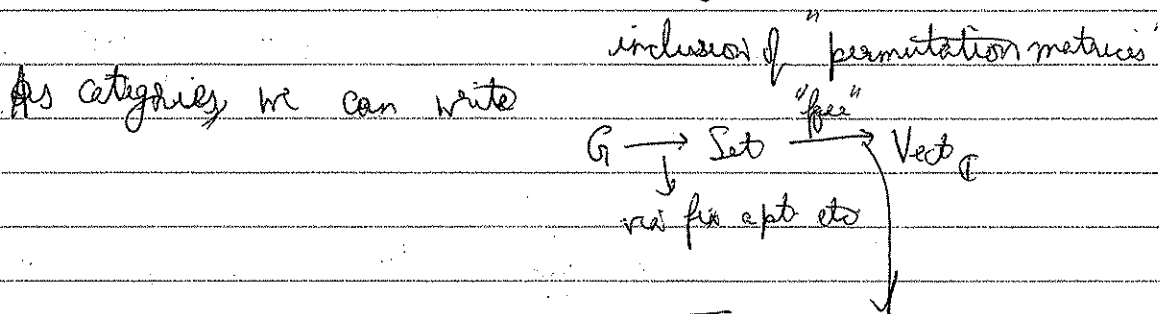
4/OCT/2007 / THU

(J. Dolan)

Switching gears Back to group theory

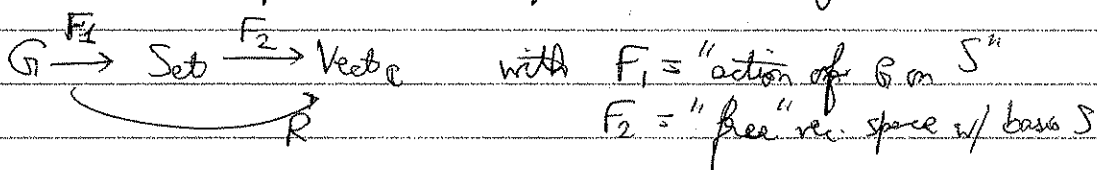
Let  $G \subseteq S!$  To get a group rep. of  $G$  from this action,

Consider  $G \subseteq \text{Mat } S$ , via  $G \subseteq S! \hookrightarrow \text{Mat}_S(\mathbb{C})$



So we get a gp. rep from a gp action  $S \mapsto V_S = \mathbb{C}^S$

Theorem: Let  $G \subseteq S!$  be a finitary transformation group (i.e. both  $G$  and  $S$  are finite). Let  $R$  be the complex representations of  $G$  obtained by



Then the Hom-Space  $\text{Hom}_G(R, R)$  is a complex v.s. with basis given by the orbits of  $G$  acting on  $S^2$ .

Note:  $\text{Hom}_G(R, R)$  is a little ambiguous. Here we're looking at the category of complex representations of  $G$ , so  $\text{Hom}_G(R, R)$  are the "G-equivariant" linear operators on  $R$ , thought of as the vector spaces they represent. So this is a vector space of linear operators.

(i.e. commutes with the  $G$ -action)

Proof / Sketch (by example)  $S = \{A, B, C\}$ ,  $G = \{\text{id}, (AC)\}$

Then  $R = \langle A, B, e \rangle \cong \mathbb{C}^3$ ,  $G = \{\text{id}_3, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \sigma\}$

So we need matrices commuting with  $\sigma$

Solving, this means matrices of the form  $\begin{pmatrix} D & E & F \\ G & H & G \\ F & E & D \end{pmatrix}$

This is a finite-dimensional vector space with elements

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

P                      Q                      R                      S                      T

Each of these relates to an orbit of  $G$  acting on  $S^2$ , eg

$$\begin{array}{|c|} \hline A \\ \hline A \\ \hline \end{array} \leftrightarrow \begin{array}{|c|} \hline C \\ \hline C \\ \hline \end{array} \quad \text{is } P$$

$$\begin{array}{|c|} \hline A \\ \hline C \\ \hline \end{array} \leftrightarrow \begin{array}{|c|} \hline C \\ \hline A \\ \hline \end{array} \quad \text{is } Q$$

$$\begin{array}{|c|} \hline B \\ \hline B \\ \hline \end{array} \quad \text{is } R$$

$$\begin{array}{|c|} \hline B \\ \hline C \\ \hline \end{array} \leftrightarrow \begin{array}{|c|} \hline B \\ \hline A \\ \hline \end{array} \quad \text{is } T$$

$$\begin{array}{|c|} \hline A \\ \hline B \\ \hline \end{array} \leftrightarrow \begin{array}{|c|} \hline C \\ \hline B \\ \hline \end{array} \quad \text{is } S$$