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$\boxed{\text{Switching gears}}$ Back to group theory

Let $G \subseteq S!$ To get a group rep. of G from this action,

Consider $G \subseteq \mathrm{Mat}\, S$, via $G \subseteq S! \hookrightarrow \mathrm{Mat}_S(\mathbb{C})$

inclusion of "permutation matrices"

As categories we can write

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\text{via fix a pt etc}"'] & \mathrm{Set} \arrow[r, "\text{"free"}"] & \mathrm{Vect}_{\mathbb{C}} \end{tikzcd}$$

So we get a gp. rep from a gp action $S \mapsto V_S = \text{"}\mathbb{C}^S\text{"}$

$\boxed{\text{Theorem}}$: Let $G \subseteq S!$ be a finitary transformation group (ie both G and S are finite). Let R be the complex representation of G obtained by

$$\begin{tikzcd} G \arrow[r, "F_1"] \arrow[rr, bend right, "R"'] & \mathrm{Set} \arrow[r, "F_2"] & \mathrm{Vect}_{\mathbb{C}} \end{tikzcd}$$

with F_1 = "action of G on S"
F_2 = "free" vec. space w/ basis S

Then the Hom-Space $\mathrm{Hom}_G(R, R)$ is a complex v.s. with basis given by the orbits of G acting on S^2.

$\boxed{\text{Note}}$: $\mathrm{Hom}_G(R,R)$ is a little ambiguous. Here we're looking at the category of complex representations of G, so $\mathrm{Hom}_G(R,R)$ are the "G-equivariant" linear operators on R, thought of as the vector spaces they represent. So this is a vector space of linear operators.

[ie, commutes with the G-action]

Proof/ Sketch (by example) $S = \{A, B, C\}$, $G = \{id, (AC)\}$

Then $R = \langle A, B, C \rangle = \mathbb{C}^3$, $G = \left\{ id_3, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \sigma \right\}$

So we need matrices commuting with σ

Solving, this means matrices of the form $\begin{pmatrix} D & E & F \\ G & H & G \\ F & E & D \end{pmatrix}$

This is a finite-dimensional vector space with elements

$$\underset{P}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}, \underset{Q}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}, \underset{R}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}, \underset{S}{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}, \underset{T}{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}$$

Each of these relates to an orbit of G acting on S^2, eg

$\boxed{\frac{A}{A}} \longleftrightarrow \boxed{\frac{C}{C}}$ is P

$\boxed{\frac{A}{C}} \longleftrightarrow \boxed{\frac{C}{A}}$ is Q

$\boxed{\frac{B}{B}}$ is R

$\boxed{\frac{B}{C}} \longleftrightarrow \boxed{\frac{B}{A}}$ is T

$\boxed{\frac{A}{B}} \longleftrightarrow \boxed{\frac{C}{B}}$ is S