10 Oct 2007

[Ask JB for explanations of the 2 "Remarks" from his previous lecture (see notes!).]

(1)

(a) As remarked in an earlier lecture, the idea here is ~~to~~ try and categorify integers, so that a formula eg. a summation, is converted into eg. a set being identified as being partitioned into some "special" subsets.

(b) eg. ~~$|D(n)|$~~ Fix an uncombed Young diagram D in $[n]$.

Then

$$\begin{array}{ccc} |D(n)| = \binom{n}{n_1, \dots, n_k} \in \mathbb{N} & \xleftarrow[\lim_{q\to 1}]{} & |D(\mathbb{F}_q^n)| = \binom{n}{n_1, \dots, n_k}_q \\ \uparrow {\scriptstyle \text{decategorification, or cardinality } |\cdot|} & & \uparrow {\scriptstyle |\cdot|} \\ D(n) \in \text{Finite Sets} & \underset{\text{"}\lim_{q\to 1}\text{" ?}}{\dashleftarrow} & D(\mathbb{F}_q^n) = \text{Product of Grassmannians} \end{array}$$

Several things are missing from this picture:

(a) Is there a valid map, where one does a "base change"?

(b) Is there such a map, so that the above picture commutes?

(c) Does it even make sense to speak of a commuting picture here?

(d) <u>Even more basic</u>: What families do these belong to?

(c) Well, $\binom{n}{n_1, \dots, n_k} \in \mathbb{N}$

$\binom{n}{n_1, \dots, n_k}_q \in \mathbb{Z}(q)$.

$D(n) \in$ Finite Sets
$D(\mathbb{F}_q^n) \in$?? Proj Var's $/\mathbb{F}_q$? Motives ? No one knows!

But this picture is actually better than it looks! Because

$\boxed{\text{Lemma}}$ $\binom{n}{n_1, \dots, n_k}_q \in \mathbb{Z}_{\geq 0}[q]$!

The justification behind this → answers one question raised last week (Remark (1))
→ uses Schubert cells.

and NOTE: because $\binom{\text{multinom}}{}$ = $\prod$ binomials,

hence it suffices to show that $\binom{n}{k}_q \in \mathbb{Z}_{\geq 0}[q]$

(d) "Justification" — easy example: $D = \boxed{}\boxed{}$ $n = 3$

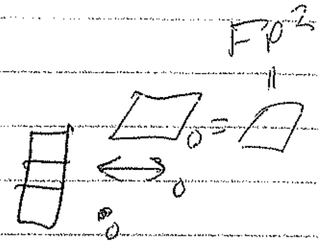
$\Rightarrow D(F^3) = FP^2$ = lines in F^3
(= points in FP^2)

Then $D(F^3) = \dfrac{F^3 \setminus \{0\}}{F \setminus \{0\}} = 1 + F + F^2$. How?

$\boxed{\text{Recall}}$ $FP^{n-1} = FP^{n-2} \amalg F^{n-1}$ because this is the hyperplane @ ∞, etc.

eg. lines in F^2 = F $\amalg$ = pt @ ∞

So now what we're basically doing is:

Pick a particular ~~total~~ flag $D_0 =$ 

A D-flag could relate to this via

$\bullet = \circ_0 \quad = \{\infty\}$

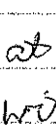
$\bullet \neq \circ_0, \quad \bullet \in \longleftrightarrow_0 \quad = FP^1 = F \amalg \{\infty\}$

$\bullet \notin \longleftrightarrow_0, \quad \bullet \in \square_0 \quad = FP^2 = F^2 \amalg FP^1$

This gives us a ~~the~~ decomposition of FP^2 into three Bruhat classes $1, F, F^2$.

② $\boxed{\text{Bruhat classes in general}}$

ⓐ A flag variety $D(F^n)$ is the ~~adjoint~~ disjoint union of Bruhat classes, obtained as follows:

[Write $D_0 =$ 
Choose a D_0-flag $=$ total flag Φ], and now we look at how a D-flag "interacts / intersects" with this fixed D_0-flag Φ.

ⓑ This is why Jim introduced the Lemma last time:

$$\mathrm{Free}(\mathrm{Orbit}(X \times Y)) \cong \mathrm{Hom}_G(\mathrm{Free}(X), \mathrm{Free}(Y))$$

Here, $G = GL(F^n)$.

Now write $D(F^n) \times D_0(F^n) = \amalg B_v$ as a disjoint union of " G-orbits.

Next, write $D(F^n) = \coprod_i \{\phi : (\phi, \Phi) \in B_i\}$

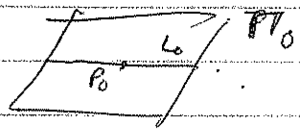
($\Phi = D_0$ - total flag)

and ~~these~~ components are the Bruhat classes.

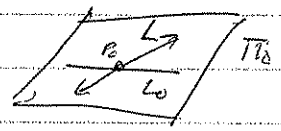
ⓒ The Schubert cells are the closures of ↓, and hence are (disjoint) unions of Bruhat classes.

e.g. $D =$ , $D_0 =$ , $\Phi = (P_0, L_0, \Pi_0)$

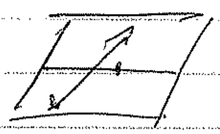
$D = (L)$

(i) 

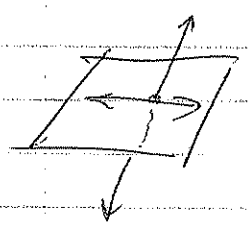
$L = L_0$ $\quad$ 1

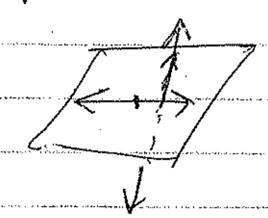
(ii) 

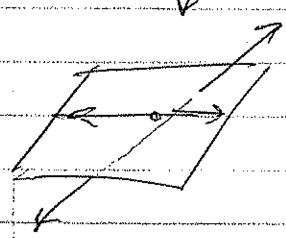
$P_0 \subseteq L \subseteq \Pi_0$ (and no better) $\quad$ F (slopes)

(iii) 
$L \subseteq \Pi_0$ (a n b) $\quad$ F^2 (slope $= F$, intercept $= F$)

OR


$P_0 \subseteq L$ (a n b) $\quad$ F^2 (slopes)

(iv) 
$\exists P' : P' \subseteq L$, $P' \subseteq L_0$ (a n b) $\quad$ F^3 (F^2 = slopes, F = inte.)

(v) 
(a n b) $\quad$ F^4 (F^2 = slope, F^2 = intercept)

(d) So, $D(F^4) \cong 1+F+2F^2+F^3+F^4$

$$\Rightarrow D(\mathbb{F}_q^4) = \binom{4}{2}_q = 1+q+2q^2+q^3+q^4$$

That it is in $\mathbb{N}[q]$ is not obvious from the defn!

But we can explicitly compute it in base 'q' — and check that it agrees w/ above!

$$\binom{4}{2}_q = \frac{\cancel{1\cdot 11}\cdot 111\cdot 1111}{\cancel{1\cdot 11}\cdot 1\cdot 11} = \frac{111\cdot 1111}{11} = 111\cdot 101$$

$$= 11211$$

$$= q^4+q^3+2q^2+q+1$$

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(3) We'll see that $\binom{n}{k}_q \in \mathbb{N}[q]$ using Bruhat classes.

But first, we show that $\binom{n}{k}_q$ is palindromic

(because then every $\binom{n}{n_1,\dots,n_k}_q$ is, ~~too~~!)

Pf To see this, one has to replace q by q^{-1} and see if one gets the same thing (modulo a power of q).

$$\frac{(q^n-1)(q-1)}{(q^k-1)(q^{n-k}-1)} \longrightarrow \frac{(q^{-n}-1)(q^{-1}-1)}{(q^{-k}-1)(q^{k-n}-1)}$$

$$= \frac{(1-q^{-n})(1-q^{-1})}{(1-\ )(1-\ )} = \left(\frac{(q^n-1)(q-1)}{(q^k-1)(q^{n-k}-1)}\right) q^{-1}$$

④ How to show each $\binom{n}{k}_q$ is in $\mathbb{Z}_{\geq 0}[q]$?

$\downarrow$

One can actually do ② © using Reduced Row Echelon Forms!

ⓐ And this kind of thing would work in general!

This gives a Bruhat class decomposition for any Grassmanian

This "proves" that $\binom{n}{k}_q \in \mathbb{Z}_{\geq 0}[q]$

This proves it for $\binom{n}{n_1, \dots, n_k}_q$ as well!

(FAILED APPROACH)

ⓑ $\boxed{\text{Aliter}}$ Prove the following "Pascal-Type Formula":

$$\binom{n+1}{k}_q = -q\,[k-1]_q \binom{n}{k}_q + [n-k+2]_q \binom{n}{k-1}_q$$

$\boxed{\text{Pf}}$ Expand the RHS to get

$$-q\,[k-1]_q \frac{[n]_q!}{[k]_q!\,[n-k]_q!} + [n-k+2]_q \frac{[n]_q!}{[k-1]_q!\,[n-k+1]_q!}$$

$$= \frac{[n]_q!}{[k]_q!\,[n-k+1]_q!}\left(-q\,[k-1]_q\,[n-k+1]_q + [n-k+2]_q\,[k]_q\right)$$

and the term in the parentheses is

$$-q\,\frac{(q^{k-1}-1)(q^{n-k+1}-1) + (q^{n-k+2}-1)(q^{k}-1)}{(q-1)^2}$$

$$= \frac{-q\left(q^{n} - q^{k-1} - \cancel{q^{n-k+1}} + 1\right) + \left(q^{n+2} - \cancel{q^{n-k+2}} - q^{k} + 1\right)}{(q-1)^2}$$

$$= \frac{q^{n+2} - q^{n+1} - q + 1}{q-1} = \frac{(q^{n+1}-1)(q-1)}{(q-1)^2} = [n+1]_q$$

whence the entire RHS now becomes

$$\frac{[n]_q!}{[k]_q!\,[n-k+1]_q!}\,[n+1]_q = \binom{n+1}{k}_q$$

(c) Once this formula is done, ~~the last is easy by induction.~~ ~~We know~~ $\cancel{\binom{n}{k}_q \in \mathbb{N}}$

So this is NOT the right formula! But something similar should do. For instance, wikipedia tells us:

$$\binom{n}{k}_q = q^{k}\binom{n-1}{k}_q + \binom{n-1}{k-1}_q$$

Let's check this! Starting as usual, with the RHS:

$$q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q = \frac{[n-1]_q!}{[k]_q!\,[n-k]_q!}\left(q^k [n-k]_q + [k]_q\right)$$

$$= \frac{q^k\left(q^{n-k} - \cancel{1}\right) + \left(\cancel{q^k} - 1\right)}{q-1} = \frac{q^n - 1}{q-1} = [n]_q$$

$$\Rightarrow \text{RHS} = \frac{[n-1]_q!}{[k]_q!\,[n-k]_q!}\,[n]_q = \binom{n}{k}_q$$

—X—

So once this claim is done, one uses the "base case"

$$\binom{n}{0}_q = 1 \quad \forall\, n \geq 0$$

to show that the $\binom{n}{k}_q$'s are, indeed, in $\mathbb{N}[q]$

by induction on n, k!

(Proved).

(And the categorical/geometric meaning is, as was said ~~today~~, in terms of Bruhat classes and Schubert cells.)

(5) Note: That the q-binomials are palindromic, comes from thinking about these as Hilbert polynomials of compact real manifolds (i.e. of their cohomology).

Then, that these polynomials are symmetric/palindromic, is but what we better know, as <u>Poincaré duality</u>!