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Today's theme: Use Hecke operators as a tool / application of Geometry/Logic, in understanding Representation Theory.

(1) **Example**: G = finite group, R = doubly transitive permutation rep. of G

(a) **Lemma** Then $R = \mathbb{C} \oplus M$ as G-modules, with $\mathbb{C} = \mathbb{C}^G$ and M a G-irrep too

Defn (a) R = perm rep of G means $\exists$ Set S s.t

$R = \text{Free}(S)$, $G \leq S!$, $(|S| > 1)$.

(b) R = Doubly transitive perm rep $\Leftrightarrow$ G acts on S^2, and $\exists!$ 2-orbits Δ_S, $(S^2 \setminus \Delta_S)$.

G acts ~~on~~ trans. on each of these.

(b) The proof uses Hecke operators, and Schur's Lemma (+ Maschke's Thm) as follows:

"The irreps of a finite group G, form an orthonormal basis for the 2-Hilbert space of fin dim G-reps."

One attaches formal coeff's to reps, and then makes $(G\text{-rep})^{fd}$ into a ring. For inner products, one uses

$\langle V, W \rangle := \text{Hom}_G(V, W)$. $\left(= \text{Span}/\text{Set of Hecke operators} \right)$ = a vector space.

One can extend ~~$\mathcal{B}$~~ so that linearity works here.

This makes the category nice enough to be made into a ~~2~~ Hilbert space. But because it ~~seems to be~~ much more – namely, a categorification of some sort! – we call it a 2 – Hilbert space.

But then, the irreps behave nicely w.r.t. Hom's, ie

$$\dim \mathrm{Hom}(V,V') = \delta_{V,V'} \quad (V,V' \in \hat{G} = G\text{-irreps})$$

and this is what Schur's Lemma says.

$\boxed{\text{Note}}$: The use of the word "basis" is because of Maschke Theorem (which says: $(\mathbb{C}G\text{-rep})^{fd}$ is a semisimple category)

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(2) Back to our example: This is immediate from the ~~thm~~ result from a previous class:

(a) $\mathrm{Hom}_G(R,R)$ has a basis given by the G-orbits in: But by double transitivity, this # = 2

So $\mathrm{Hom}_G(R,R) = 2$. By Schur + Maschke, if

$$R = \bigoplus_{V_i \in \hat{G}} n_i V_i, \text{ then } \langle R,R \rangle = \sum n_i^2$$

So it must be 1^2+1^2, ie $\sum n_i = 1+1 = 2$ ▨

(B) Moreover, the diagonal subset $\Delta_S \subseteq S^2$ will correspond to the trivial representation, if we use the proof of the result of ~~that~~ previous class.

— X —

(3) We'll do a "more powerful application" now: we'll "find" ~~or~~ "characterise" all the irreps of (4!).

(a) By the above lemma, we're trying to find some basis (o.n.) of a Hilbert space. We'll use (the categorified version of) "Gram-Schmidt Orthonormalization".

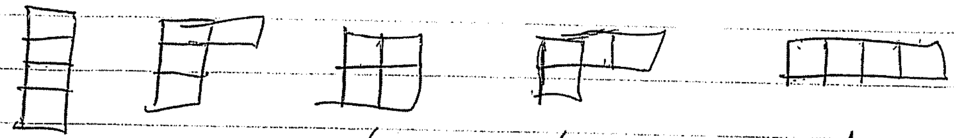
So, we need to start with some basis and go on.

(b) $\boxed{\text{MIRACLE}}$ Why this categorified version? Because then we work with actual objects in our category, and (hopefully) end up with actual objects ~~of~~ in an ~~o~~ orthonormal basis.

Note that in any old Hilbert space, there are ∞'ly many orthonormal bases. But in our setup, we only have one basis that ~~a~~ also consists of objects in our Category! (Called a "categorical basis")

So, a priori, we shouldn't expect our use of the Gram-Schmidt process to yield a categorical basis. But, miraculously, it does!

(c) Back to our example: We look at the flag-reps. of S_4 (defined earlier). These are given (by'ly) by



dim = 24   dim = 12   6   4   1 $\dim = \binom{n}{n_1, \dots, n_k}$

(I) There are not irred., but as JB/JD had said in an earlier class, we can find ~~an~~ irreps (which are *also* indexed by these Young tableaux) inside the corresponding flag-rep.!

(II) Moreover, we start with this basis and carry out Gram-Schmidt. How did we know the above ~~basis~~ was a basis?

Easy answer: There isn't, but go ahead anyways! If we end up with nonzero elts at the end, then we know we're good!
(And it will be thus :) !)

(Actual answer: needs math ...)

But first, let's write down the $\langle \, , \rangle$ matrix:

24	12	6	4	1
	7	4	3	1
		3	2	1
			2	1
				1

and the matrix is symmetric in general b/c

$$\operatorname{Hom}_G(V,W) \cong \operatorname{Hom}_G(W^*,V^*)$$

$$\cong$$

$$\left(\operatorname{Hom}_G(W,V)\right)^*$$

So dim's are the same!