23/OCT/2007/TUE

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The classical and quantum Pascal's Triangles

(1) (a) Write the coefficients of $x^k y^{n-k}$ in the n^{th} row.

$n=0$ 1

$n=1$ 1 1

$n=2$ 1 2 1

$n=3$ 1 3 3 1

⋮

and this shows us ~~the~~ a recursion relation that the $\binom{n}{k}$'s satisfy:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

(b) But of course, we'll try to categorify this. So:

LHS = k-elt subsets of n

RHS = (2 choices) = $\{1 \in k\text{-elt set} \Rightarrow \binom{n-1}{k-1}\}$

$\{1 \notin k\text{-elt set} \Rightarrow \binom{n-1}{k}\}$

(re, pick a distinguished elt. $1 \in [n]$,..).

(c) Now, let's q-deform all ~~this~~. How does one get a relation between quantum binomial coefficients?

One can compute, and see that $\binom{n}{k}_q$ does NOT equal $\binom{n-1}{k}_q + \binom{n-1}{k-1}_q$. (So $0<k<n$).

Here's how to categorify: So $\binom{n}{k}_q$ = # k-dim subspaces of $\mathbb{F}_q^n$.

Let's in fact work over a general field F.

So, to use $(n-1)$ here, we fix a $(n-1)$-dim'nal subspace S_{n-k}, and consider our k-subspace S_k in relation to this hyperspace.

One option is that $S_k \subseteq S_{n-1}$. Otherwise, we claim ~~that~~ $S_k \cap S_{n-1}$ ~~has~~ dim $k-1$. Because:

$S_k \cap F^n$ has dim k, and $\dim(S_k \cap S_{n-1}) \neq k$, so

$S_k \cap S_{n-1} = (k-1)$-dim.

And how many such degrees of freedom for the extra degree of freedom (of $S_k / S_k \cap S_{n-1}$) are there?

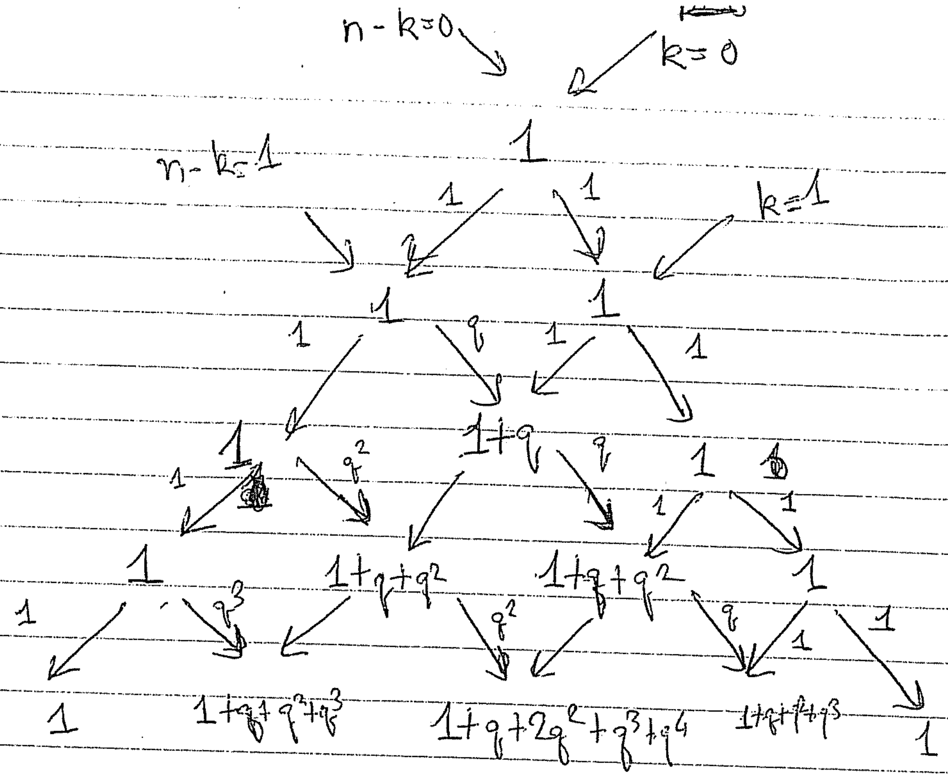
It can be from anything in ~~the~~ remaining $n-k = (n-1)-(k-1)$ degrees of freedom!

$$\text{So, } \left[\binom{n}{k}_F \cong \binom{n-1}{k}_F + F^{n-k} \times \binom{n-1}{k-1}_F \right.$$
$$\left. \text{in the category of sets.} \right]$$

(d) De-categorifying: $F \to \mathbb{F}_q$, so we now get the

q-Pascal identity: $\boxed{\binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q}$

One now writes the q-Pascal triangle:



ⓔ <u>Observations</u>: ① Coeff's are $\in \mathbb{N}_0[q]$, b/c of the q-Pascal identity!
(This is the other proof, not using Row Echelon Forms)

② There are two symmetries involved:

ⓐ Each entry is palindromic (which can be shown by replacing q by q^{-1} etc — & by Poincaré duality, in the setup of Grassmannians and Bruhat/Schubert cells!)

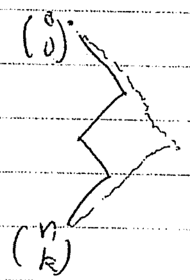
ⓑ The entire picture is symmetric, i.e. each row is "palindromic". Which will lead to a proof that q-binomial and q-multinom coeffs are "symmetric" in their arguments. We'll see ~~this~~ next time!

③ Just as any row in the classical Pascal Δ adds up to 2^n, or if we attach $x^k y^{n-k}$, to $(x+y)^n$, what is the q-analog, or the q-binomial theorem?

This is the question that will occupy us for the rest of the lecture.

② ⓐ If we drop a ball from the top, and it chooses left/right at each stage with probability $1/2$, then we get a "Gaussian" dist below at the base.

More specifically, the # ways to get from $\binom{0}{0}$ to $\binom{n}{k}$ = $\binom{n}{k}$!



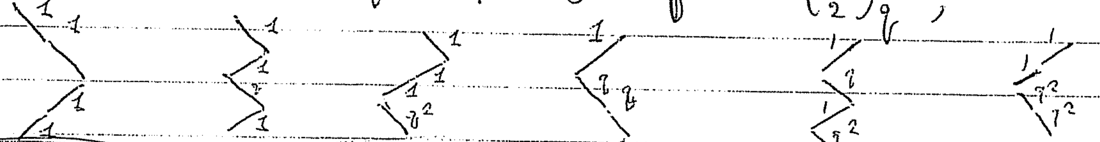
$\boxed{\text{Pf}}$ From $(0,0)$, have to go left $n-k$ paths/steps & right k steps

and we can do either at each stage!

So ✓.

ⓑ How does this q-generalize? Say we're looking at paths again. Then we weight each path by not 1, but a power of q! e.g. for $\binom{4}{2}_q$,

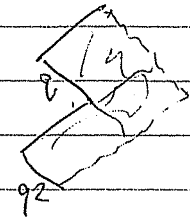
path



weight 1 $\quad q \quad q^2 \quad q^2 \quad q^3 \quad q^4$

Other than this complicated method of multiplying all "numbers written on lines" to get the weight, what's a more "seeable" method?

Well, each such path determines an area to the right! eg.

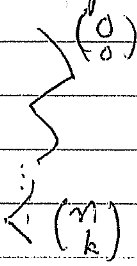


$= 3$ units / blocks

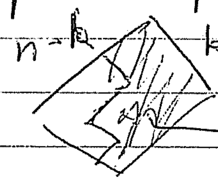
Hence weight $= q^3$!

(c) Let's make this completely specified now.

Clearly, for



, we have to work with all possible paths in an

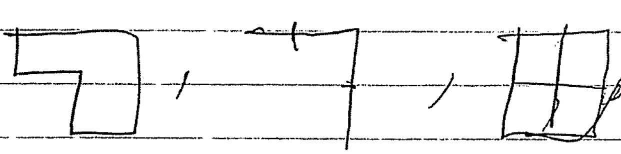


$(n-k) \times k$ - rectangle

and now every path (weighted) creates a Combed Young diagram! Which ones?

Defn Let $\Omega_{n,k} = \{$ Combed Young diagrams with $\le k$ columns & $\le n-k$ rows $\}$

eg.



$\in \Omega_{4,2}$

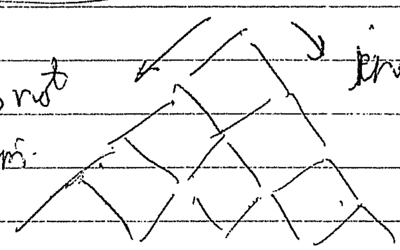
etc.

Prop^n Then $|\Omega_{n,k}| = \binom{n}{k}$, and

$$\binom{n}{k}_q = \sum_{\omega \in \Omega_{n,k}} q^{|\omega|}$$, where $|\omega| = \#$ boxes in ω

(2) Another approach is to use Bruhat cells and $\cap$ with ~~fla~~ a pre-fixed complete flag $F_\bullet$ of $\mathbb{F}_q^n$.
We have any fixed k-dim subsp. F_k ~~flag F~~, and $F_\bullet$. Eventually, ~~F reaches~~ $F_\bullet$ reaches $\mathbb{F}_q^n$, so $F_{\bullet i} \cap F_k$ reaches F_k.

and we're seeing when the dimensions start to $\uparrow$



Moreover, the YD you start out with (from the chosen path) is exactly the shape that ~~fp~~ shows up in the associated row echelon form, as the undetermined "$*$"'s in it!