25/OCT

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(1)

(a) Continuing from last ~~time~~ week: We were trying to do categorified Gram-Schmidt process, and ended up with a matrix of Hom-dim's (= dim's of (Hecke operators) spaces) -

1	1	1	1	1
1	2	2	3	4
1	2	3	4	6
1	3	4	7	12
1	4	6	12	24

(b) We'll change bases from this one → which is flag rep's → to irreducible rep's, which will be indexed by the same symbols (with different meanings). This ambiguity is "sort of" justified because in each copy of the flag rep, there's a unique copy of the associated irrep in it.

This will make the "change of basis matrix" a lower $\triangle^r$ one, with 1's on the diagonal

$$
\begin{array}{c}
 & \text{OLD} \\
\text{NEW} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 \\ -1 & 2 & 1 & -3 & 1 \end{bmatrix}
\end{array} = A.
$$

(c) How does its inverse look? Also lower Δ, also unipotent, also integer entries. But more importantly, they're all ≥ 0 !!

$$\begin{array}{cc} & \text{NEW} \\ \text{OLD} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & 3 & 2 & 3 & 1 \end{pmatrix} = B \end{array}$$

(d) $\boxed{\text{Obsns}}$ How to compute dim of irreps?

(Irrep Ξ e.n. basis b/c irrep $\Leftrightarrow$ (dim End( ) = 1) by Schur + Maschke)

So, dim of original basis is $\underline{d} = (1, 4, 6, 12, 24)^T$

and hence dim of irreps is $A \cdot \underline{d} = (1, 3, 2, 3, 1)^T$!

$\underline{\text{So}}$ $\rightarrow$ the last row in $B = A^{-1}$ turns out to be !! (*)

(e) In part; the original flag reps were "lin. indep", b/c Gram Schmidt does yield a valid basis!

(2) (a) Now we want to categorify this, so that we actually look at the representation (irrep.) inside the flag representation.

(b) Moreover, we're doing some calculations for 4! here — but they work for $GL(4)$ as well $\rightarrow$ because as $q \rightarrow 1$, $GL_4(\mathbb{F}_q) \rightarrow S_4 = 4!$

(*) The reg. rep. has (dim X) copies of the irrep X, $\forall X$

(c) In any case, we now want to categorify all this.
So we'll devise some useful and suggestive notation.

eg. 7 in the  ×  is to be

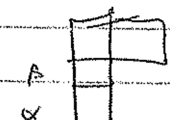
set of matrices w/ entries in $\mathbb{N}_0$, so that row & column sums are

$$\begin{array}{c|ccc} & 2 & 1 & 1 \\ \hline 2 & & & \\ 1 & & & \\ 1 & & & \end{array}$$

So there's 7 of them:

$$\underset{(a)}{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \quad \underset{(b)}{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

But  = ordered pairs (α, β) | ~~So how do~~

So how do we describe the way 2 ordered pairs interact / overlap.

eg. (b) is that they are the same.

(a) is: (α, β) and (β, α). etc.