$\boxed{9.27.07}$ "Geometric Representation Theory"

- Goals: relationships between QM & classical mechanics or classical logic.

Groups & symmetries:

what do groups act on

Classical	Quantum mechanics
Sets	Vector Spaces
Function	Linear Operators

$\underbrace{\text{Sets, Function}}_{\text{category Set}}$

symmetries

group is a category 1 element

$G \xrightarrow{R} \text{Set}$ $\quad$ $\cdot \circlearrowright g$ $\quad$ $R(\cdot) \circlearrowleft R(g)$

$G \xrightarrow{R} \text{Vect}$ $\quad$ $\cdot \circlearrowright g$ $\quad$ $R(\cdot) \circlearrowright R(g)$

g is morphism,
R is a functor

We want to study representations from their actions: { diff between rep & action

$$\begin{tikzcd} G \arrow[r] \arrow[dr] & \text{Set} \arrow[d] \\ & \text{Vect} \end{tikzcd}$$

→ We want this.

One functor is the "free". Given set $X \longmapsto \mathbb{C}^X$
or $n \longmapsto \mathbb{C}^n$ (n elt set $\mapsto$ n dim vector space)

• 2 examples:

(1) The symmetric group S_n is a main example (group of all automorphisms). We will call $S_n = n!$

$n!$ acts on n, hence $\mathbb{C}^n$

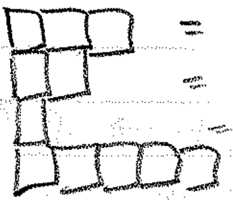
say $3!$ acts on $\mathbb{C}^3$: $\mathbb{C}^3 \cong \{(x,x,x)\} \oplus \mathbb{C}^2$
(These representations are not irreducible)

The idea is that this won't necessarily give irreducible reps. This trick always can apply

If G acts on X & $S(X)$ is a set of structures on X, then G acts on $S(X)$

We will ∴ get lots of actions (∴ representations) of $n!$ from structures on n-elt sets.

Where do we get structures? Young Diagrams
An uncombed n-box Young Diagram is a list $n_1, \dots n_k$ of positive $\mathbb{Z}$ with $n_1 + n_2 + \dots + n_k = n$

$n_1 =$ 
$n_1 = 3$
$n_2 = 2$
$n_3 = 1$
$n_4 = 5$

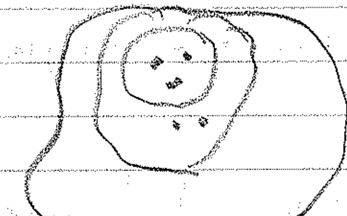
uncombed 11-box Young Diagram

If $n_i \geq n_{i+1}$ then it's combed (or just Young)

Given an uncombed n-box Young diagram D we define a <u>D-flag</u> on an n-elt set X as:

$$\emptyset = X_0 = X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots \subseteq X_k = X$$

where $|X_i - X_{i-1}| = n_i$ in the Young diagram.



just like a partition of n elt set.

Suppose D =  . Then there are 3! D-flags

on 3. example: $\begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline 2 \\ \hline \end{array}$ so there is a 6 dimensional

rep of 3!

<u>Thm:</u> Every irrep of n! is a subrep of the representation of n! on $\mathbb{C}^{D(n)}$ ($D(n)$ is set of D-flags on the n elt set) For some uncombed Young diagrams.

In fact it is enough to use combed ones.
In fact there is a one to one correspondence between irreps of n! and n-box Young diagrams.

Example #2:

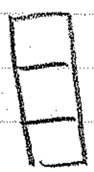
For any Field F $GL(n,F)$ has a rep on F^n (the vector space). F^n is still a set

There is a finite field with q elts F_q for each prime power $q=p^n$ $n \geq 1$.

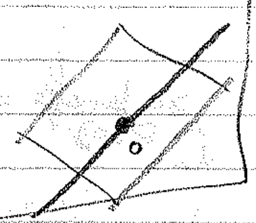
Now let D be an uncurled n-box Young diagram.
A D-flag on F^n is:

$$\{0\} = X_0 = X_1 \subseteq X_2 \subseteq \dots \subseteq X_k = F^n \text{ where}$$

X_k is a nested subspace s.t. $\dim(X_i / X_{i-1}) = n_i$

example → take $D =$  How many D-flags are there on F_q^3 (infinite if F is not finite)

this is a D-flag on $\mathbb{R}^3$



How many lines thru origin?

answer: $\dfrac{q^3 \overset{\text{minus origin}}{-1}}{\underset{\text{overcounting}}{q-1}}$ $\quad |F_q^3| = q^3$

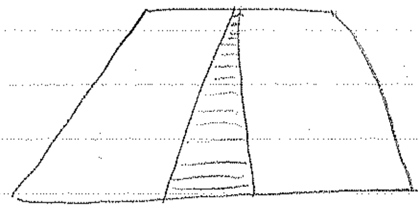
How many planes? Pick a line in F_q^3/line

q-Deformation $\left\{ \underbrace{\dfrac{q^3-1}{q-1}}_{\text{converges to 3 as } q\to 1} \cdot \underbrace{\dfrac{q^2-1}{q-1}}_{2 \text{ as } q \to 1} \cdot \underbrace{\dfrac{q-1}{q-1}}_{1} = [3]_q! \right.$

$= 3!$

This suggests that a set of n-elts is a n-dim vector space over a 1elt Field.

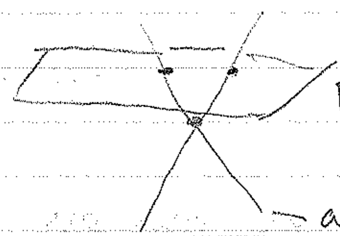
- Projective Geometry:



parallel lines meet at horizon (∞ far away)

Any 2 pts meet at a line ; any ~~[illegible]~~ line is defined by 2 pts

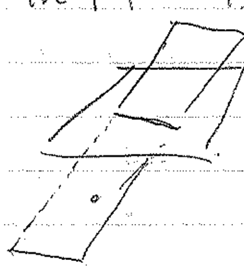
$\mathbb{R}P^2 = \{1 \text{ dim subspaces of } \mathbb{R}^3\}$ real projective plane



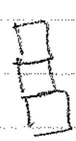
→ the lines that don't intersect the plane correspond to pts of ∞ (form a circle with antipodal pts identified)

$\therefore \mathbb{R}P^2 = \mathbb{R}^2 \cup \mathbb{R}P^1$

$\therefore$ For any field F we define $FP^{n-1} = \{1d \text{ subspaces of } F^n\}$
A line in FP^{n-1} is a 2 dim subspace of F^n



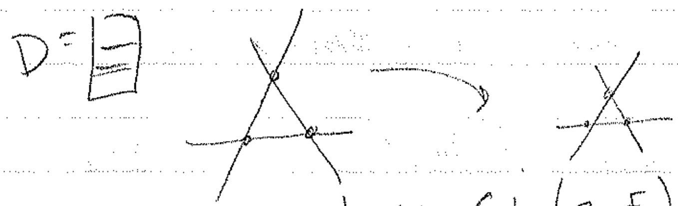
A D-flag in F^n: $\{0\} = X_0 \subseteq X_1 \subseteq \dots \subseteq X_k = F^n$
Each X_i can be thought of a figure in projective geometry:

D =  what object in FP^2
(a point, a line, the plane)

(6)

So $GL(n,F)$ acts on F^n, hence it acts on $D(F^n)$ where D is any n-box unsorted Young diagram.

We get a representation of $GL(n,F)$ on $\mathbb{C}^{D(F^n)}$. But we don't get all irreps of $GL(n,F_q)$ from this..



$3! \leftrightarrow GL(3,F)$

- # of lines thru origin:

  not origin $\rightarrow$ # of pts: q^3-1. pick pt. other pts are scalar multiples $\{0,\dots,q-1\}$ (characteristic q) $\therefore$ $q-1$ pts (not including 0)

  $\therefore$ # lines thru origin is $\dfrac{q^3-1}{q-1}$

- # planes containing line:

  pick a line now pick a pt of the line. That defines the plane.

  # of pts not on line q^3-q

  # of pts on the plane not on the line $q^2-q=q(q-1)$

  $$\therefore \frac{q(q^2-1)}{q(q-1)} = \frac{q^2-1}{q-1}$$

- # of "total spaces": $\dfrac{q-1}{q-1}$ $\leftarrow$ # of pts not in a plane; $\leftarrow$ # of pts on line containing that pt