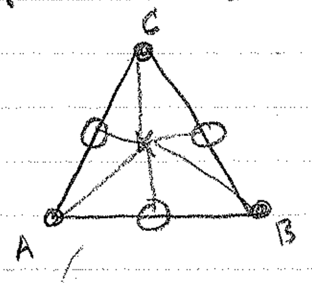
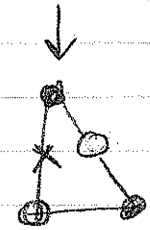
$\boxed{10.4.07}$

X ≡ red
O ≡ blue
● ≡ green

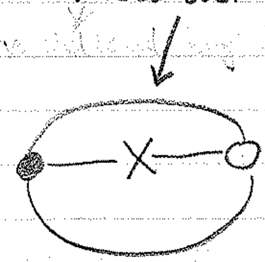
• orbi-simplex for 3!:



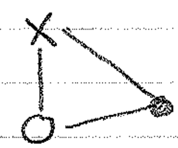
mod out by subgroup that switches 2 corners: (A, B)



mod out by C_3

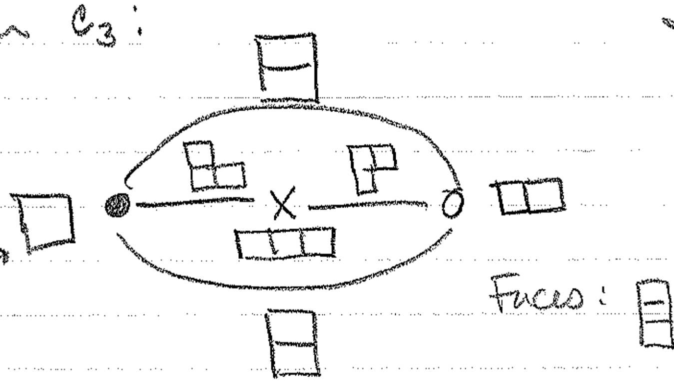


mod out by entire group:



These pictures tell you about the axiomatic theory that describes the structure preserved by 3!

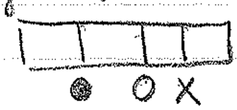
• Consider C_3:



represents all of the set

Faces:

* the rule for assigning Young diagrams:

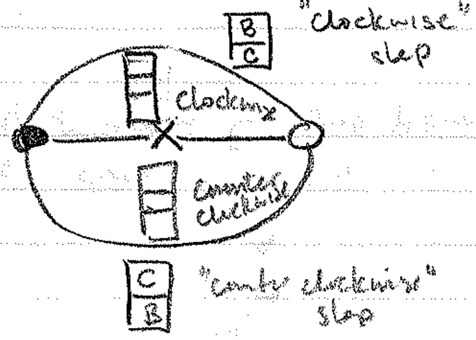


# of rows − 1 = dim of each piece.

each piece of the orbi-simplex of a transformation group $G \subseteq S!$ represents a G-orbit of flags on S of the type corresponding to the Young diagram label.

(2)

Intuitively C_3 preserves "orientation":



We are most interested in the vertical Young diagrams.

Axiomatic description of this theory:

1. Binary predicates "CW" (clockwise)
   $CW(x,y)$ "clockwise step from x to y",
   and "CCW"
2. We claim also we have equality (x,x) (not obvious)
3. We have an axiom:

$$\forall x,y\ \big(CW(x,y) \vee CCW(x,y) \vee (x=y) \big)$$

(actually could use exclusive or)

Every vertical Young diagram becomes a predicate in the theory:

 } binary predicate

 } 3 predicate

What does this have to do with group representation theory?

- Transformation groups : group rep's:
  Let $G \subseteq S!$ be a transformation group.

To get a rep for G, we should turn S (our set G operates on) into a vector space.

$\therefore G \subseteq Mat_S$ (S×S matrices). $S! \hookrightarrow Mat_S(\mathbb{C})$
where inclusion is of the permutation matrices.
(This is a free construction on S)

$$\begin{tikzcd} G \arrow[r] \arrow[rr, bend right, "\text{representation}"'] & SET \arrow[r, "\text{free}"'] & VSP_{\mathbb{C}} \end{tikzcd}$$

Now we reinterpret the orbi-simplex picture to tell us something about the representation.

- Thm: Let $G \subseteq S!$ be finitary transformation group (i.e. G & S are finite). Let R be the complex representation of G obtained by:

$$\begin{tikzcd} G \arrow[r] \arrow[rr, bend right, "R"'] & Set \arrow[r] & VSP_{\mathbb{C}} \end{tikzcd}$$

R is the "Vspace" and the Functor.

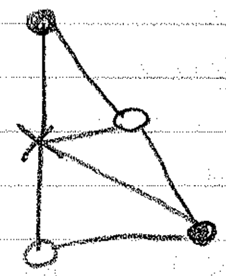
Then the Hom-space, $Hom_G(R,R)$ is a complex vector space with basis given by the orbits of G acting on S^2. (The orbits of G act on S^2 are pieces $\boxminus$ from our diagrams.)

S^2 are ordered pairs of S

$Hom_G(R,R)$: R is an object in the category of complex rep's of G, and this is the set of morphisms from R to itself.
(Equivariant operators)

- "Proof" by example:

2 elt subgroup of 3! i.e.:



Let $S = \{A, B, C\}$; $G = \left\{ \begin{matrix} ABC \\ ABC \end{matrix} , \begin{matrix} ABC \\ CBA \end{matrix} \right\}$

R is a vector space of dim 3 with basis ABC.
i.e. $R = \langle A, B, C \rangle$ (free span of ABC)

$$\begin{array}{c|c} & \begin{matrix} A & B & C \end{matrix} \\ \begin{matrix} A \\ B \\ C \end{matrix} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

is the matrix that represents our non-trivial grp elt.

$\boxed{\text{The morphisms in } \mathrm{Hom}_G(R,R) \text{ are all the matrices that commute with group action:}}$

$$\begin{array}{c|c} & \begin{matrix} A & B & C \end{matrix} \\ \begin{matrix} A \\ B \\ C \end{matrix} & \begin{bmatrix} D & E & F \\ G & H & J \\ K & L & M \end{bmatrix} \end{array} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} F & E & D \\ J & H & G \\ M & L & K \end{bmatrix}$$

$$\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \begin{bmatrix} D & E & F \\ G & H & J \\ K & L & M \end{bmatrix} = \begin{bmatrix} K & L & M \\ G & H & J \\ D & E & F \end{bmatrix}$$

These equations imply:

$$\begin{bmatrix} D & E & F \\ G & H & G \\ F & E & D \end{bmatrix} \Bigg\} \begin{matrix} \text{5 variables} \\ \therefore \text{ 5 dimension} \\ \text{vector space} \end{matrix}$$

Hence a basis is: $\begin{bmatrix} 1 & & \\ & & \\ & & 1 \end{bmatrix}$, $\begin{bmatrix} & & 1 \\ & & \\ 1 & & \end{bmatrix}$,

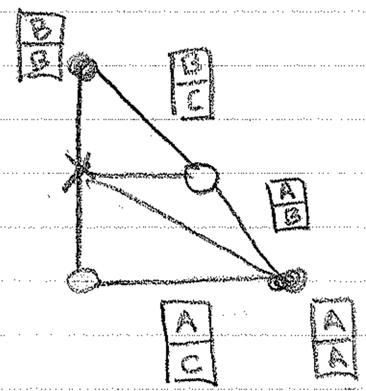
$\begin{bmatrix} & 1 & \\ & & \\ & 1 & \end{bmatrix}$, $\begin{bmatrix} & & \\ 1 & & 1 \\ & & \end{bmatrix}$, $\begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix}$

These should correspond to the orbits acting on S^2.

The matrix "is" S^2

$$\begin{array}{c|ccc} & A & B & C \\ \hline A & 1 & & \\ B & & & \\ C & & & 1 \end{array}$$

A, A gets sent to C, C by the non-trivial elt of the group



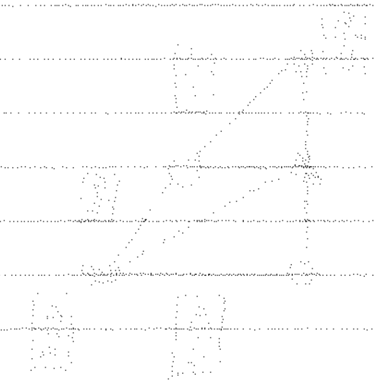
S set of D flags in elt set

~~Relay~~: think of $n!$ acting on S rather than the original set

Find Hecke operators on DFlags (n-box)

~~Happy~~ on N

[illegible]



[illegible]