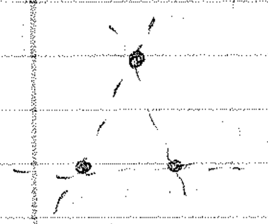
$\boxed{10.9.07}$

- Set theory — Projective geometry

finite sets

$n$

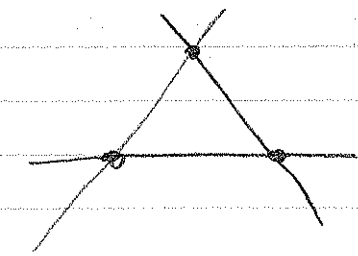


a "line" is just 2 pts in finite sets

Fin dim vector space

$F^n$ or $FP^{n-1}$

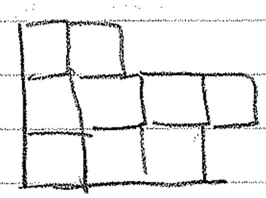
1 dim subspaces $\Longleftrightarrow$ points



$FP^2$

We pts, pairs, triples — pt, lines, planes, etc

A D flag in $n$ (the $n$ elt set) is: $\phi = X_0 \subset X_1 \subseteq \dots \subseteq X_k = n$ where

$D =$  $n_1 \; \vdots \; n_k$

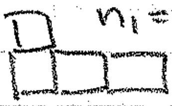
$\sum n_i = n$

$\vdots \; |X_i - X_{i-1}| = n_i$

A D flag in $F^n$ is subspaces: $\{0\} = X_0 \subseteq X_1 \subset \dots \subset X_k = F^n$ where D is again an unordered Young diagram:
$\dim(X_i / X_{i-1}) = n_i$

②

- Let's <u>count</u> the D-flags in $n$ & $F^n$ where $F = \mathbb{F}_q$.

Let $D =$  $n_1 = 1$, $n_2 = n-1$

so $|D(n)| = n$ & $|D(F^n)| = [n]_q$ ($n$th $q$-integer)

D-flags on $n$ = points of $n$

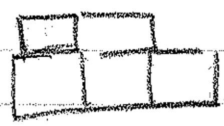
with $D(F^n)$ is a 1 dim subspace = "points" of $FP^{n-1}$ (i.e. lines thru origin in $F^n$

$$\therefore |FP^{n-1}| = \frac{q^n - 1}{q-1} = [n]_q$$

$$FP^{n-1} = \frac{F^n - \{0\}}{F - \{0\}}, \quad |FP^{n-1}| = \frac{q^n - 1}{q-1}$$

decategorify

categorify

Next let $D =$  $n_1 = 2$, $n_2 = 3$

$D(5)$ = 2-elt subsets of 5

$$\therefore |D(5)| = \binom{5}{2} = \frac{5!}{\underbrace{2!\cdot 3!}_{\text{Young subgroups - permutation of } n \text{ preserving the D flag}}} = 10$$

Group of permutation

③

$D(F^5) = \{ \text{2d subspace of } F^5 \} \cong \{ \text{line in } FP^4 \}$

idea: find a group that acts transitively on 2d subspace of $F^3$ & then mod out by a subgroup that stabilizes a plane.

$GL(5, F)$ acts transitively on $D(F^5)$ ; the stabilizer subgroup of a D-flag

$$\underbrace{\left(\begin{array}{c|c} GL(2) & \text{anything} \\ \hline 0 & \text{anything} \end{array}\right)}_{} \begin{pmatrix} * \\ * \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ * \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

5×5 matrix

this kind of subgroup is a "parabolic subgroup" of $GL(n)$ - preserves a D flag.

So to count the D-flags we count $GL(n, F)$:

$$|D(F^5)| = \frac{|GL(5,F)|}{\left| \left\{ \begin{pmatrix} GL(2) & * \\ 0 & * \end{pmatrix} \right\} \right|}$$

$$= \frac{|GL(5,F)|}{|GL_F(2) \times GL_F(3) \times F^{2\times 3}|}$$ ← set description of the matrix

$$(GL_F(2) \times GL_F(3)) \ltimes F^{2\times 3}$$ group structure (semi-direct product)

So what's $|GL(n, F_q)|$? Count column by column:

$$= |(F_q^n - \{0\}) \times (F_q^n - F_q) \times (F_q^n - F_q^2) \times \dots \times (F_q^n - F_q^{n-1})|$$

$$= (q^n - 1)(q^n - q) \dots (q^n - q^{n-1})$$

4

$$= \frac{q^n - 1}{q-1} \cdot q \cdot \frac{q^{n-1} - 1}{q-1} \cdots q^{n-1}\frac{q-1}{q-1} \cdot (q-1)^n$$

$$= [n]_q! \, (q-1)^n q^{1+2+\dots+(n-1)}$$

$$= [n]_q! \, (q-1)^n q^{\binom{n}{2}}$$

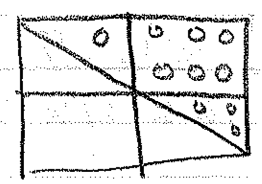
$$|D(F^5)| = \frac{|GL(5,F)|}{|GL(2)|\,|GL(3)|\,|F^{2\times 3}|}$$

$$= \frac{[5]_q! \, \cancel{(q-1)^5} \, q^{\binom{5}{2}}}{[2]_q! \, \cancel{(q-1)^2} \, \cancel{(q-1)^3} \, q^{\binom{2}{2}} q^{\binom{3}{2}} q^6}$$

$$= \frac{[5]_q!}{[2]_q! \, [3]_q!} \cdot \underbrace{\frac{q^{\binom{5}{2}}}{q^{\binom{2}{2}} q^{\binom{3}{2}} q^6}}_{1}$$

$$= \binom{5}{2}_q$$

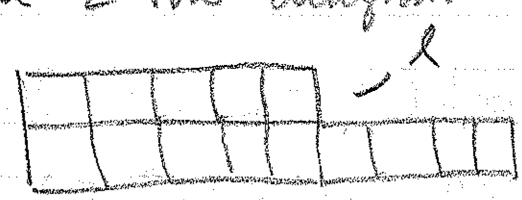
q binomial coeffic.



triangle number

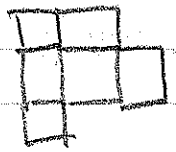
Claim: $\binom{5}{2} = \binom{3}{2} + \binom{2}{2} + 2\cdot 3$

If D is a 2 row diagram

D = 

$\therefore D(F^n)$ = Grassmanian of $l$ dim subspaces of $F^n$ = $Gr(n,l)$

$$= \binom{n}{l}_F$$

- Now let $D =$  $2 = n_1$ (a "general" Young diagram)
$3 = n_2$
$1 = n_3$

& $n = 6$

Count $D$-flags on 6: $D(6) = \binom{6}{2}\binom{4}{3}$

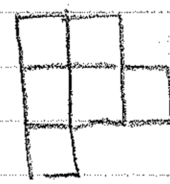
Choose 2 out of 6 & out of the remaining 4 choose 3

$$D(6) \cong \frac{6!}{2!\,4!} \cdot \frac{4!}{3!\,1!} = \frac{6!}{2!\,3!\,1!}$$

$$\cong \frac{S_6}{\underbrace{S_2 \times S_3 \times S_1}_{\text{Young subgroup preserving the } D\text{-flag}}}$$

$\therefore$ in general $|D(n)| = \dfrac{n!}{n_1! \times \cdots \times n_k!}$ } multinomial coefficient

$$= \binom{n}{n_1, n_2, \ldots, n_k}$$

<u>Finally</u>: $D(F^6)$ with $D =$ 

space, base space, fiber

$$D(F^6) = \underbrace{\binom{6}{2}_F}_{Gr(6,2)} \text{"}\times\text{"} \binom{4}{3}_F$$

$$|D(F^6)| = \binom{6}{2}_q \binom{4}{3}_q = \frac{[6]_q!}{[2]_q!\,[4]_q!} \cdot \frac{[4]_q!}{[3]_q!\,[1]_q!}$$

6

$$= \binom{6}{2,3,1}_q$$ "q-multinomial coeff"

Generally For any n-box unnumbered Young D:

$$D(F^n) = \binom{n}{n_1, n_2, \ldots, n_k}_F$$

$$\therefore |D(F^n)| = \frac{[n]!_q}{[n_1]_q! \cdots [n_k]_q!}$$

We categorified : q-deformed multinomial coeff.

- $[n]_q$ are integers to proj geom over Finite Field like integers in set theory.

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \ldots + q^{n-1}$$

categorify

$$\downarrow$$

$$[n]_F = FP^{n-1} = \frac{F^n - \{0\}}{F - \{0\}} \cong \underbrace{1 + F + \ldots + F^{n-1}}_{\text{Schubert cells.}}$$