$\boxed{10.11.07}$ (JIM DOLAN)

- Thm: Given Finite group G, a G equivariant linear operator between permutation representations is isomorphic to linear combinations of G-orbits on Cartesian product of G-sets.

a vector space

$$\left\{ \begin{array}{l} \text{G-Equivariant} \\ \text{Linear operator} \\ \text{between perm.} \\ \text{representations} \end{array} \right. \cong \left. \begin{array}{l} \text{G-orbit} \\ \text{Cartesian product} \\ \text{of G-sets} \end{array} \right\} \begin{array}{l} \text{basis} \\ \text{for vector} \\ \text{space} \end{array}$$

these are "Hecke operators" — "Geometrico-logical" relationships between types of geometric figures

There is some relation between Hecke operators & Hecke algebras

We will see a relationship between Flags (Baez's context) and geometric figures

- Example: Let G = isometries of a cube (distance preserving maps). This is a 48 elt group (rotations & reflections).

Let the 1st G-set be the "corners of the cube" (8) & let the 2nd G-set be the "edges" (12) & Note that both sets are transitive.

What are the relationships between corners & edges?

(2)

(relationships)

#1. Corner lies on edge.

#2. Corner and edge lies on same Face. (and no better)

#3. corner and edge lie on same cube (i.e. generic) (and no closer)

3 edges satisfy #1, 6 edges satisfy #2, 3 edges satisfy #3. These relationships are the G-orbits

(What about orientation-preserving isometries of a cube?)

- How does composition of the operators on the LHS of the isomorphism relate to the geometric relationships on the RHS?

→ We want to transitively "glue" relations together $X, Y \circ Y, Z \to X, Z$ (We need to work only with the atomic relations)

Atomic relationship ①: corner lies on edge
atomic relationship ②: edge lies on Face

yields corner lies on face

Consider:

$$\begin{tikzcd} \text{corner} \arrow[r, "\text{corner lies on edge}"] & \text{edge} \arrow[r, "\text{edge containing corner}"] & \text{corner} \\ X \arrow[rr, bend right, "\text{compose:}"'] & Y & Z \end{tikzcd}$$

$\Rightarrow$ corner X = corner Z

Superposition of amplitudes (quantum logic)
vs. $|\psi|^2$

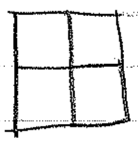
<u>or</u> $\Rightarrow$ corners X and Z lie on same edge

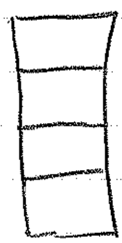
* So here we have a basis for an algebra, and multiplying the basis elts doesn't yield another basis elt.

The <u>or</u> operator here is like a "superposition"

(Like some weird linear algebra on the truth values)

- 2^{nd} <u>Example</u>: Fix 2 Young diagrams D_1 & D_2

$D_1 =$  so $G = GL(4, \mathbb{F}_q)$
and the 1^{st} G-set = D_1-flags on $\mathbb{F}_q^4$

& $D_2 =$  $\therefore$ the second G-set = D_2 flags on $\mathbb{F}_q^4$

so D_1 is corresponding to 2 dim subspaces of $\mathbb{F}_q^4$ (i.e. planes) or "lines" in projective space

& D_2 is corresponding to "a pt, on a line, in a plane in the 3-space" but <u>projectively</u>

4

Revelations between D_1 & D_2:

P

$L \qquad\qquad\qquad L'$

$\underbrace{P\,L}_{D_2} \qquad\qquad \underbrace{\quad}_{D_1}$

#1 : $L' = L$

#2 : $P \leq L' \leq PL$

#3 A: $P \leq L'$

#3 B: $L' \leq PL$

#4 : "L' touches L"

#5 : Generic

$\leq$ means "contain"

("no hotter" is implied