$\boxed{10.16.07}$ (JOHN BAEZ)

- From last time:
  - Jim is talking about how Hecke operators are invariant relationships on G-sets
  - John is talking about q-deformed multinomial coeff...
  - relationships?

- Fix unlabeled Young diagram D with n-boxes:

(For a finite set)

$$\begin{tikzcd} |D(n)| = \binom{n}{n_1,\dots,n_K} \in \mathbb{N} \arrow[d, bend left, "\text{decategorify (take cardinality)}"] \\ D(n) \in \mathrm{FinSet} \arrow[u, bend left, "\text{categorify}"] \end{tikzcd}$$

Now for fin. dim vectorspaces $\mathbb{F}_q^n$:

$$\begin{tikzcd} |D(\mathbb{F}_q^n)| = \binom{n}{n_1,\dots,n_K}_q \arrow[d, "\lim_{q\to 1}"] & D(\mathbb{F}_q^n) \arrow[l, bend right, "1:1 \text{ decategorify}"'] \arrow[d, dashed, "?"] \\ |D(n)| \arrow[r] & D(n) \in \mathrm{FinSet} \end{tikzcd}$$

(Flag variety)

what is the limit process implied by $- - \to$?

What is the field with 1 element

We know $|D(n)| = \binom{n}{n_1 \dots n_k} \in \mathbb{N}$

but what is the membership of $|D(\mathbb{F}_q^n)|$?

We know that $\binom{n}{n_1 \dots n_k}_q \in \mathbb{Z}(q)$

rational functions over $\mathbb{Z}$

What about membership of $D(\mathbb{F}_q^n)$?

Should it be in the family of projective algebraic varieties? Motives? (Nobody knows...)

Thm: Clearly $\binom{n}{n_1 \dots n_k}_q = \frac{[n]_q!}{[n_1]_q! \cdots [n_k]_q!} \in \mathbb{Z}(q)$

but in fact $\binom{n}{n_1 \dots n_k}_q \in \mathbb{N}[q]$ ← polynomial

- Easy example:

Let D =  n = 3

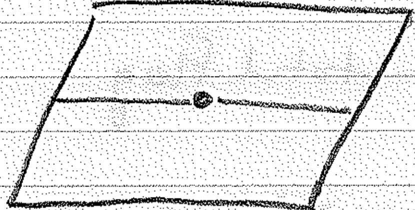
$$D(\mathbb{F}_q^3) = \mathbb{F}_q P^2 = \frac{\mathbb{F}_q^3 - \{0\}}{\mathbb{F}_q - \{0\}}$$

$$|D(\mathbb{F}_q^3)| = \frac{q^3 - 1}{q - 1} = 1 + q + q^2$$

rational

actually polynomial

since $\mathbb{F}_q P^2 = 1 + \mathbb{F}_q + \mathbb{F}_q^2$

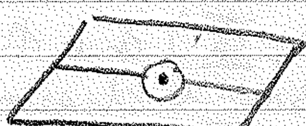


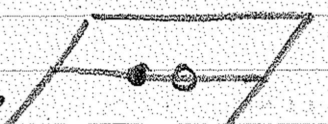
FP^2

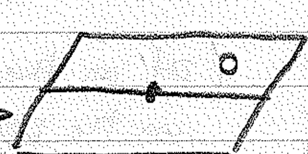
Pick a total Flag
D_0-flag with

$D_0 =$ 

A D-flag could relate to this D_0 via:

1 elt  pt is the pt.

F elts  pt is on the line

F^2 elts  pt is on the plane

$$FP^2 \cong \underbrace{1 + F + F^2}$$

these are Bruhat classes

(there closure are Schubert cells)

* So a Flag variety $D(F^n)$ is the disjoint union of Bruhat classes obtained by choosing a total flag - (D_0-flag) with

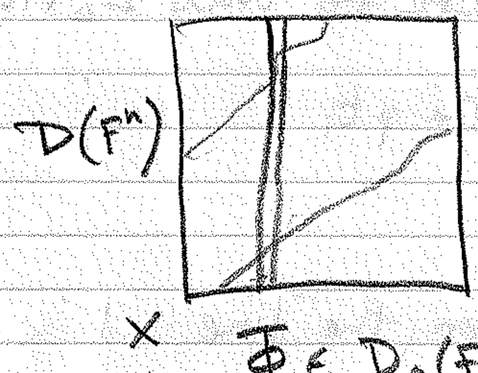
$D_0 =$  . Write $D(F^n) \times D_0(F^n) = \bigcup_i B_i$

where B_i are $GL(n)$ orbits.

Next choose a particular total flag $\Phi \in D_0(F^n)$ & write

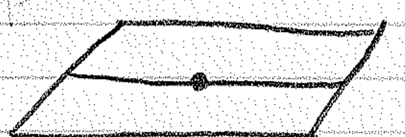
$$D(F^n) = \bigcup \{\phi : (\phi, \Phi) \in B_i\}$$

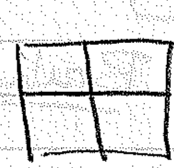
ϕ is some D-flag with relation to Φ



(ϕ, Φ) are the Bruhat classes.

- Example: $n = 4$ $D_0 =$  (pt on the line on the plane in the space)



and let $D =$  the D flags are lines in $FP3$

So pick a particular total flag $\Phi = (\underset{\text{pt}}{p_0}, \underset{\text{line}}{L_0}, \underset{\text{plane}}{\Pi_0})$

so $L = L_0$ (the set of lines that are the line L_0 are a Bruhat class)

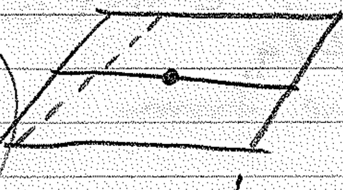
(one element) $\therefore$ $\underline{1}$

$P_0 \subseteq L \subseteq \Pi_0$

line goes thru pt & no better

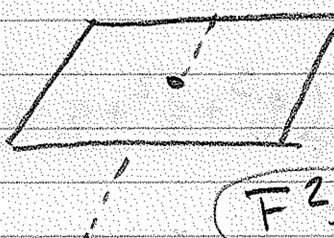
(the Schubert cell would be w/o the "no better" i.e. closure)

F elements



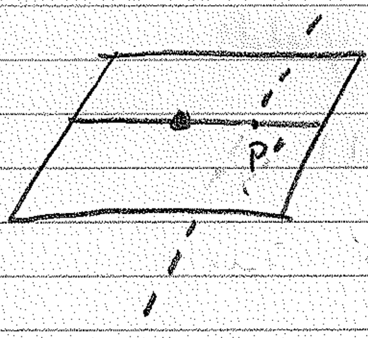
$L \subseteq \Pi_0$ (a.n.b)

F^2 elements (pt on slope)



$P_0 \subseteq L$ (a.n.b)

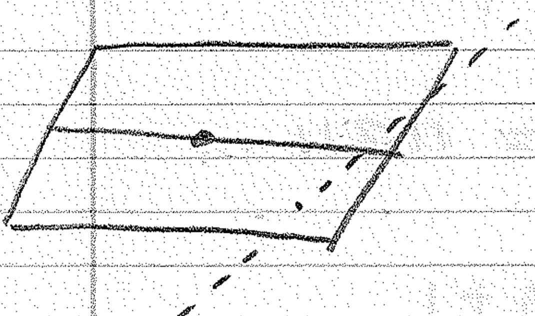
F^2 elements



p'

$\exists P',\ P' \subseteq L_0\ \&\ P' \subseteq L$ (a.n.b)

F^3 elements (pick p', pick slope, pick slope)



(a.n.b)

F^4 elements (pick 2 pts & 2 slopes)

$$D(F^4) \cong 1 + F + 2F^2 + F^3 + F^4$$

$$|D(F^4)| = \binom{4}{2}_q = 1 + q + 2q^2 + q^3 + q^4$$

* So this is showing why $\binom{4}{2}_q \in \mathbb{N}[q]$ but not obvious why this is true from definition of $\binom{4}{2}_q$ ...

$$\binom{4}{2}_q = \frac{1\,(q+1)\,(q^2+q+1)\,(q^3+q^2+q+1)}{(q+1)\,(q+1)}$$

In base q this calculation yields

$$\binom{4}{2}_q = \frac{1 \cdot \cancel{11} \cdot 111 \cdot 1111}{\cancel{11} \cdot 11}$$

$$\begin{array}{r} 1111 \\ 111 \\ \hline 123321 \end{array}$$

$\longrightarrow$ always palindrome

$$\therefore \binom{4}{2}_q = \frac{123321}{11} = 11211$$

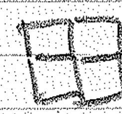
$$= q^4 + q^3 + 2q^2 + q + 1$$

Key Idea → IF our polynomial is in $\mathbb{N}[q]$ then it's a palindromic polynomial.

- Why for any Grassmanian do our Bruhat classes look like F^k?

i.e. lets show $|D(F^n)| \in \mathbb{N}[q]$ when D has 2 rows e.g.  k, $n-k$ then $D(F^n)$ is a Grassmanian of k-planes in F^n.

* We can use some linear algebra tricks ... (row eschelon form)

For our $n=4$, $D=$  example: $\begin{cases} P_0 = \langle (1,0,0,0) \rangle \\ L_0 = \langle (1,0,0,0), (0,1,0,0) \rangle \\ \Pi_0 = \langle (1,0,0,0), (0,1,0,0), \ldots \rangle \end{cases}$

$1: \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ matrix with a choosen basis in affine space

$F: \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{pmatrix}$ here $*$ is any number in F.

$F^2: \begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \end{pmatrix}$

$F^2: \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{pmatrix}$

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$$F^3: \begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{pmatrix}$$

$$F^4: \begin{pmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix}$$