Geometric Representation Theory

CLASSICAL	QUANTUM
Sets	Vector Spaces
Functions	Linear Operators

$$\begin{tikzcd} \cdot \arrow[loop below, "g"] & G \arrow[r, "R"] & \mathrm{Set} & R(\cdot) \arrow[loop right, "R(g)"] \end{tikzcd}$$

$$\begin{tikzcd} \cdot \arrow[loop below, "g"] & G \arrow[r, "R"] & \mathrm{Vect} & R(\cdot) \arrow[loop right, "R(g)"] \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r] \arrow[dr, bend right] & \mathrm{Set} \arrow[d] & X \arrow[d, mapsto] & n \arrow[d, mapsto] \\ & \mathrm{Vect} & \mathbb{C}^X & \mathbb{C}^n \end{tikzcd}$$

2 examples:

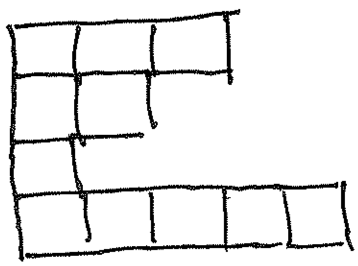
1) The symmetric group $S_n = n!$ acts on n, hence on $\mathbb{C}^n$.

$3!$ acts on $\mathbb{C}^3$.

$$\mathbb{C}^3 \cong \{(x,x,x)\} \oplus \mathbb{C}^2$$

If G acts on X & $S(X)$ is some set of structures we can put on X, then G acts on $S(X)$. We'll get lots of actions of $n!$ from structures on n-element sets.

An uncombed n-box Young diagram is a list $n_1, \ldots, n_k$ of positive integers w. $n_1 + \cdots + n_k = n$.



$n_1 = 3$
$n_2 = 2$
$n_3 = 1$
$n_4 = 5$

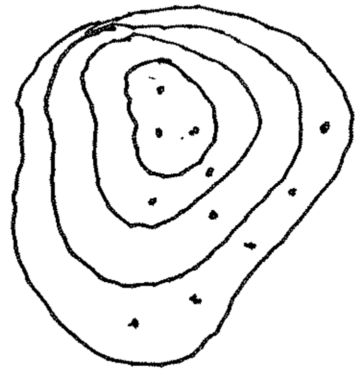
11-box Young diagram

If $n_1 \geq n_2 \geq \cdots$, we have a Young diagram.

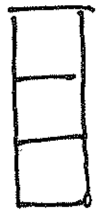
Given an uncombed n-box Young diagram D, a D-flag on an n-element set X is:

$$\emptyset = X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_k = X$$

where $|X_i - X_{i-1}| = n_i$



Suppose

D = 

Then there are 3! D-flags on 3:

Thm. – Every irrep of $n!$ is a subrep. of the rep. of $n!$ on $\mathbb{C}^{D(n)}$ for some uncombed n-box Young diagram D. In fact it's enough to use combed ones.

($D(n)$ is the set of D-flags on the n-element set.)

2) For any field F, $GL(n, F)$ has a rep. on F^n. There's a unique field with q elements, F_q, for each prime power $q = p^n$, $n \geq 1$.

Let D be an uncombed n-box Young diagram.

A <u>D-flag</u> on F^n is:

$$\{0\} = X_0 \leq X_1 \leq \cdots \leq X_k = F^n$$

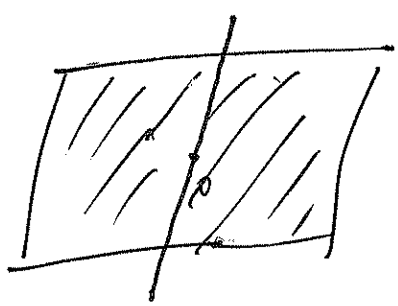
s.t.

$$\dim (X_i / X_{i-1}) = n_i .$$

Example:

$D =$ 

How many D-flags are there on $\mathbb{F}_q^3$?



# of lines through the origin:

$$\frac{q^3 - 1}{q - 1}$$

# of planes containing a chosen line:

$$\frac{q^2 - 1}{q - 1}$$

So we count D-flags using q-factorials.

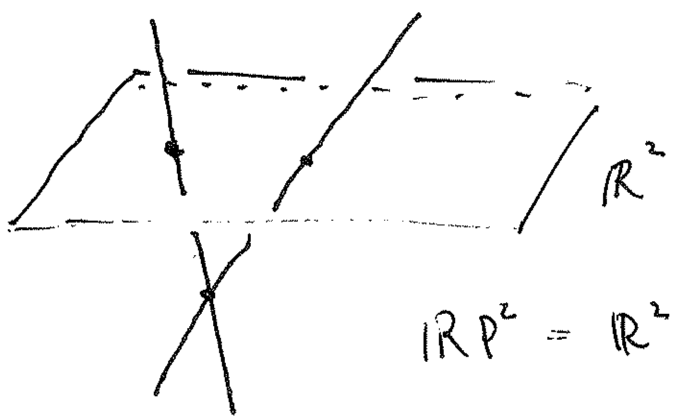
$$\frac{q^3-1}{q-1} \quad \frac{q^2-1}{q-1} \quad \frac{q-1}{q-1} = [3]_q!$$

$$\begin{array}{ccc} \| & \| & \| \\ [3]_q & [2]_q & [1]_q \end{array}$$

Exercise: Do this for any uncombed n-box Young diagram.

Projective geometry

$\mathbb{R}P^2 = \{1d \text{ subspaces of } \mathbb{R}^3\}$



$\mathbb{R}P^2 = \mathbb{R}^2 \cup \mathbb{R}P^1$

$FP^{n-1} = \{1d \text{ subspaces of } F^n\}$

A line in FP^{n-1} is a 2d subspace of F^n, etc.

A D-flag in F^n

$$\{0\} = X_0 \le X_1 \le \cdots \le X_k = F^n$$

gives a geometrical structure on FP^{n-1}.

$D =$  - gives a point on a line in FP^2.

$GL(n, F)$ acts on F^n, hence it acts on $D(F^n)$

(where D is any uncombed n-box Young diagram).

So, get a rep of $GL(n, F)$ on $\mathbb{C}^{D(F_q^n)}$

- but alas, not all irreps of $GL(n, F_q)$ lie in these.