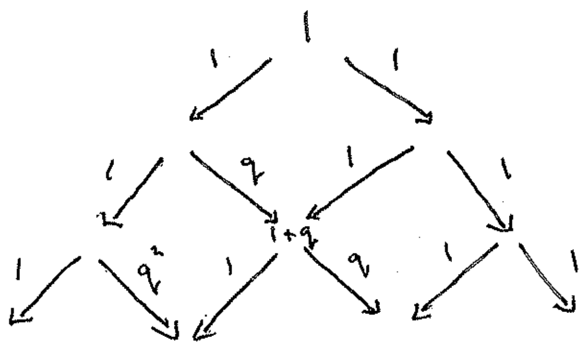
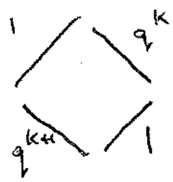
The relationship between the q-deformed Pascal's triangle and quantum groups.

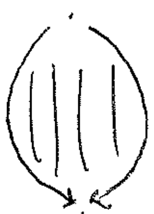


The quantum Pascal triangle with "phase" q



the different paths differ by q

this is analogous to Bohm-Aharonov effect

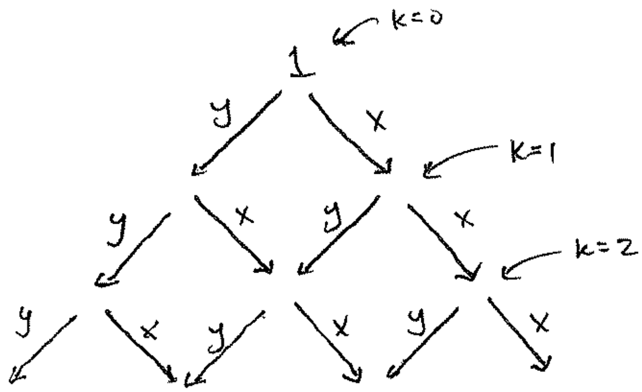


$e^{i\int B}$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

We want a formula where $\binom{n}{k}$ is replaced by $\binom{n}{k}_q$.

For the binomial formula we just needed x & y to commute. Now we want them to q-mute



$$xy = qyx$$

$$(x+y)^2 = x^2 + xy + yx + y^2$$

$$= x^2 + (1+q)yx + y^2$$

$$(x+y)^3 = x^3 + (1+q+q^2)yx^2 + (1+q+q^2)y^2x + y^3$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}_q y^k x^{n-k}$$

We're used to polynomial functions (valued in k) on the plane k^2:

$$k[x,y] = k\langle x,y\rangle / \langle xy = yx\rangle$$

but now we're thinking about the "quantum plane" — the noncommutative algebra:

$$k_q[x,y] = k\langle x,y\rangle / \langle xy = qyx\rangle$$

→ noncommutative geometry.

The group $GL(2,k)$ acts on the plane k^2 hence on $k[x,y]$. The "quantum group" $GL_q(2,k)$ "acts" on the quantum plane $k_q[x,y]$.

Algebraic geometry is about studying spaces by studying functions on those spaces.

Algebraic Geometry

Geometry	Algebra
Space X	Commutative algebra $\mathcal{O}(X)$ of functions on X
Maps $\varphi : X \to Y$	Algebra homomorphism $\varphi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ $f \longmapsto f \circ \varphi$
Group G - space w. $m : G \times G \to G$ $inv : G \to G$ $id : \underline{1} \to G$ ($\underline{1}$: terminal object) s.t. ...	Commutative Hopf algebra - $\mathcal{O}(G)$ - commutative alg. $m^* : \mathcal{O}(G) \to \mathcal{O}(G \times G) \cong \mathcal{O}(G) \otimes \mathcal{O}(G)$ $inv^* : \mathcal{O}(G) \to \mathcal{O}(G)$ $id^* : \mathcal{O}(G) \longrightarrow \mathcal{O}(\underline{1}) = k$ s.t. ...

A group G can act on a space X:

$$\alpha : G \times X \to X$$

s.t. . . .

The commutative Hopf algebra $\mathcal{O}(G)$ "coacts" on the comm. alg. $\mathcal{O}(X)$:

$$\alpha^* : \mathcal{O}(X) \to \mathcal{O}(G) \otimes \mathcal{O}(X)$$

s.t. . . .

$GL_q(2,k)$ is a (noncomm.) Hopf algebra coacting on the quantum plane $k_q[x,y]$.

To make noncomm. geometry (& q-binomial formula) less formal go back to our chart:

$$\begin{tikzcd}
\binom{n}{k} \in \mathbb{N} \arrow[r, "q\text{-deforming}"] \arrow[d, "\text{categorifying}"'] & \binom{n}{k}_q \in \mathbb{N}[q] \arrow[d, "\text{categorifying}"] \\
\binom{n}{k} \in \mathrm{FinSet} \arrow[r, "q\text{-deforming}"'] & \binom{n}{k}_{\mathbb{F}_q} \in ??
\end{tikzcd}$$

$$\binom{n}{k}_F = \{ V \subseteq F^n : V \text{ is } k\text{-dim} \}$$

$$\begin{tikzcd}
 & \binom{n}{k}_F \arrow[dl, "y"'] \arrow[dr, "x"] & \\
\binom{n+1}{k}_F \arrow[dr, "x"'] & & \binom{n+1}{k+1}_F \arrow[dl, "y"] \\
 & \binom{n+2}{k+1}_F &
\end{tikzcd}$$

x, y are relations between sets —

but they're <u>invariant</u> under action of $GL(n, F)$.

(Hecke operators)

$xy = q\,yx$ — relation between Hecke operators!