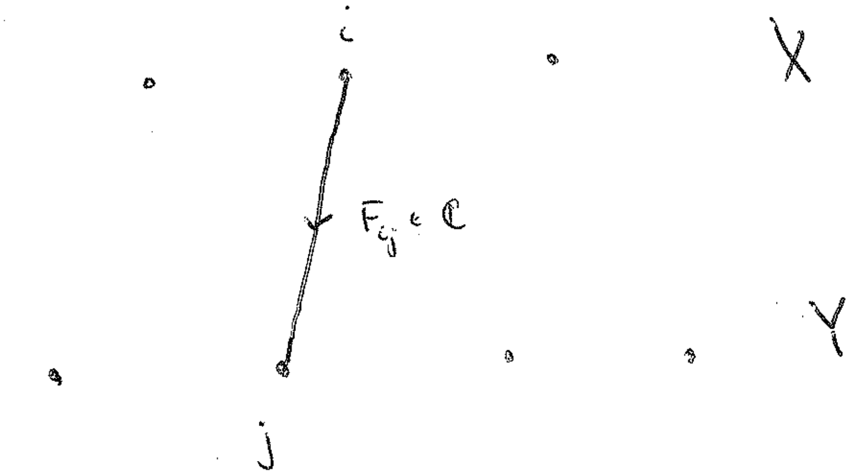
Matrix Mechanics

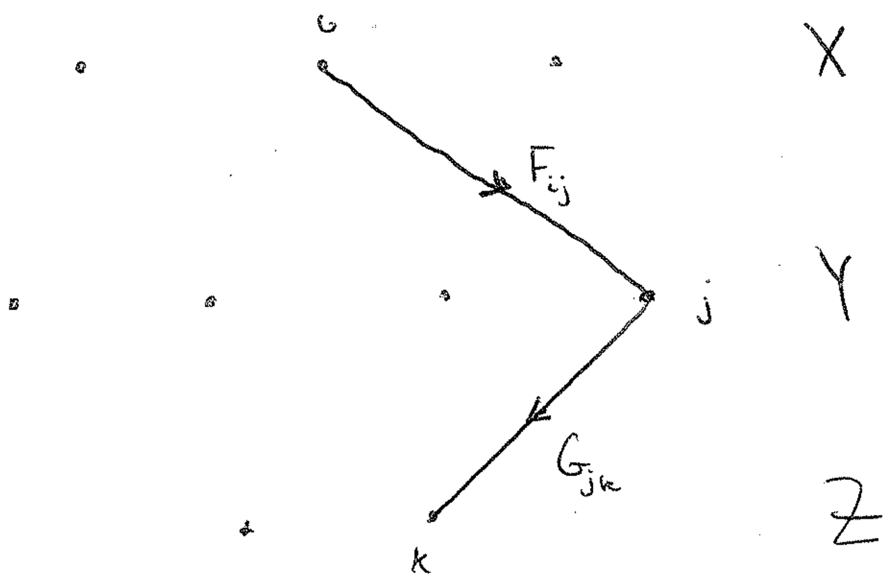
X, Y, Z finite sets



X states of a system

Every process is given an amplitude

What about composing processes?



$F: X \nrightarrow Y$

is

$F: X \times Y \to \mathbb{C}$

or linear

$F: \mathbb{C}^X \to \mathbb{C}^Y$

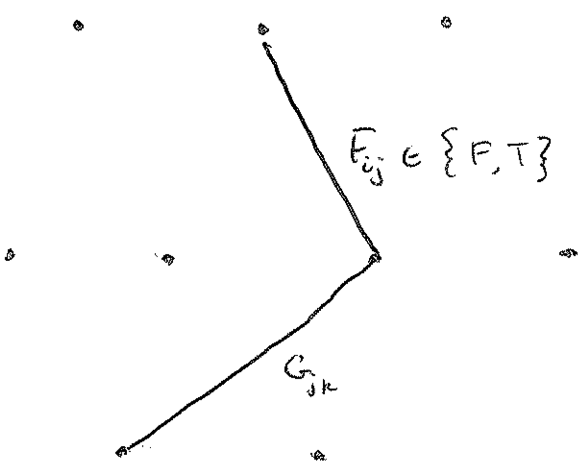
$$(FG)_{ik} = \sum_{j \in Y} F_{ij} G_{jk}$$

We can generalize from $\mathbb{C}$ to any rig.

Let's do matrix mechanics with the Boolean rig

$$\{F, T\} = \{0, 1\}$$

replacing $\mathbb{C}$.



amplitude

$\downarrow$

possibilities

$\times \to$ AND $\qquad + \to$ OR

A relation $F : X \nrightarrow Y$ is

$$F : X \times Y \longrightarrow \{0,1\} \quad \text{or} \quad \text{linear } F : \{0,1\}^X \longrightarrow \{0,1\}^Y$$

Given a relation

$$F : X \times Y \longrightarrow \{0,1\}$$

you can reinterpret it as a linear operator

$$\tilde{F} : X \times Y \longrightarrow \mathbb{C}$$

using $i : \{0,1\} \hookrightarrow \mathbb{C}$.

If G acts on X & Y, we can talk about F being G-invariant, which would imply $\tilde{F}$ is G-invariant (intertwining operator).

Do we have $\tilde{F}_1 \tilde{F}_2 = \widetilde{F_1 F_2}$?

i.e. is $\sim$ a functor?

$(\tilde{1} = 1$ o.k.)

This would mean

$$\sim : \text{FinRel} \longrightarrow \text{FinVect}$$

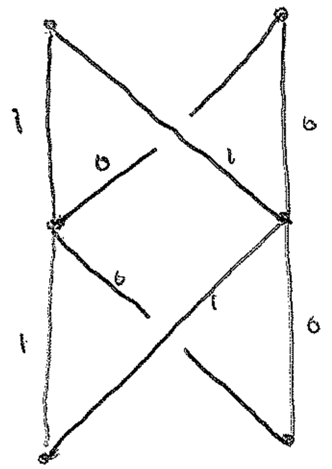
is a functor.

$i : \{0,1\} \longrightarrow \mathbb{C}$ - <u>not</u> a rig homomorphism!

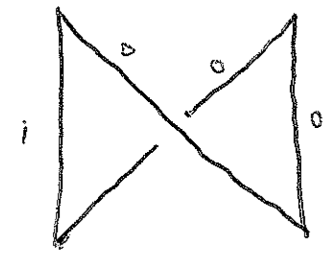
preserves $\cdot$ but not $+$.

So, No!

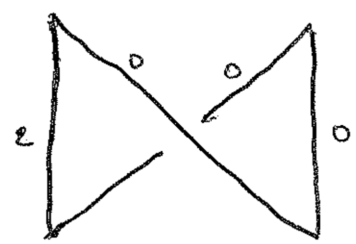
Let's see what happens when we compose.



in FinRel =



in FinVect =



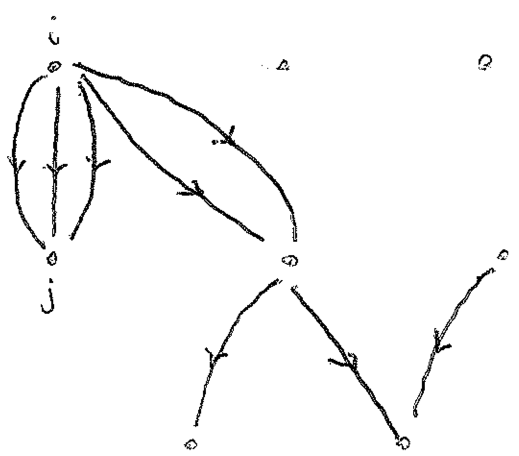
Instead of $\{0,1\}$-valued matrices, we could use $\mathbb{N}$-valued matrices, since $i : \mathbb{N} \hookrightarrow \mathbb{C}$ is a rig homomorphism.

(also $\mathbb{N} \longrightarrow \{0,1\}$ is a homomorphism)

We could also use FinSet-valued matrices — called "spans"!

$+ \longrightarrow$ coproduct

$\times \longrightarrow$ product



$$F : X \times Y \longrightarrow \mathrm{FinSet}$$

$$G : Y \times Z \longrightarrow \mathrm{FinSet}$$

$$(FG)_{ik} = \coprod_{j \in Y} F_{ij} \times G_{jk}$$

$$F : X \times Y \longrightarrow \mathrm{FinSet}$$

$$|F| : X \times Y \longrightarrow \mathbb{N}$$

$$|\tilde{F}| : X \times Y \longrightarrow \mathbb{C}$$

Now we get

$$\widetilde{|FG|} = \widetilde{|F|}\,\widetilde{|G|}$$

A matrix $F : X \times Y \longrightarrow \mathrm{FinSet}$ is also called a <u>span</u>, since:

$$\begin{tikzcd} & S = \coprod_{i \in X, j \in Y} F_{ij} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd}$$

$p_1(s) = i$

$p_2(s) = j$

knows all about F.

We can recover F_{ij} as

$$F_{ij} = \{ s \in S \mid p_1(s) = i \ \& \ p_2(s) = j \}$$

We also want to be able to compose spans

$$\begin{tikzcd}
 & & ST \arrow[dl] \arrow[dr] \arrow[ddll, bend right] \arrow[ddrr, bend left] \arrow[dd, phantom, very near start, "\vee"] & & \\
 & S \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & T \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\
X & & Y & & Z
\end{tikzcd}$$

This is a pullback.

$$ST = \{ (s,t) \in S \times T : p_2(s) = q_1(t) \}$$

We have:

- finite sets $X, Y, \ldots$
- spans of finite sets $S : X \to Y$, i.e. $\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$

  & we can compose them via pullback,

  & we have identity spans

what about associativity?

$$
\begin{tikzcd}
 & & S(TU) \arrow[ddl] \arrow[drr] & \cong & (ST)U \arrow[dll] \arrow[ddr] & & \\
 & & ST \arrow[dl] \arrow[dr] & & TU \arrow[dl] \arrow[dr] & & \\
 & S \arrow[dl] \arrow[dr] & & T \arrow[dl] \arrow[dr] & & U \arrow[dl] \arrow[dr] & \\
W & & X & & Y & & Z
\end{tikzcd}
$$

So, we get a 2-category, since associativity holds only up to isomorphism.

$$
\begin{tikzcd}
S \arrow[r, "\sim"] \arrow[d] \arrow[dr] & T \arrow[dl] \arrow[d] \\
X & Y
\end{tikzcd}
$$

A comm. diagram of this sort is an isomorphism of spans.

There is a bicategory of: finite sets
spans of finite sets
isomorphisms of spans

∴ There is a category (FinSpan) of: finite sets
isomorphism classes of spans

∴ we are getting a functor $\mathrm{FinSpan} \longrightarrow \mathrm{FinVect}$.