Matrix Mechanics

We've seen that for any rig R there's a category $\mathrm{Mat}(R)$ where:

- objects are finite sets: $X, Y, \ldots$
- a morphism $f: X \nrightarrow Y$ is a matrix $f: X \times Y \to R$
- composition is matrix multiplication.

Given a rig homomorphism $\varphi: R \to S$ we get a functor from $\mathrm{Mat}(R) \to \mathrm{Mat}(S)$:

$$\mathrm{Mat}(\varphi): \mathrm{Mat}(R) \to \mathrm{Mat}(S)$$

We have:

$$\begin{tikzcd} & \{0,1\} = \text{Booleans} \\ \mathbb{N} \arrow[ur] \arrow[dr] & \\ & \mathbb{C} \end{tikzcd}$$

This is because $\mathbb{N}$ is the initial object in the category of rigs.

$$\begin{tikzcd} & \mathrm{Mat}(\{0,1\}) \simeq \mathrm{FinRel} \arrow[d, dashed, "\sim \text{ not a functor!}"] \\ \mathrm{FinSpan} \simeq \mathrm{Mat}(\mathbb{N}) \arrow[ur] \arrow[r] & \mathrm{Mat}(\mathbb{C}) \simeq \mathrm{FinVect}_{\mathbb{C}} \end{tikzcd}$$

FinSpan is the category coming from a 2-category with:

- finite sets as objects
- spans $\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$ as morphisms $f : X \nrightarrow Y$
- $\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[drr] & S' \arrow[dl] \arrow[d] \\ X & Y \end{tikzcd}$ (commuting) as 2-morphisms

by decategorifying it - getting:

- finite sets as objects
- isomorphism classes of spans as morphisms

$$\begin{tikzcd} S \arrow[r, "\sim"] \arrow[d] \arrow[dr] & S' \arrow[dl] \arrow[d] \\ X & Y \end{tikzcd}$$ iso. of spans

We have a functor

$$\mathrm{FinSpan} \longrightarrow \mathrm{FinVect}_{\mathbb{C}}$$

& similarly

$$F : G\text{-}\mathrm{FinSpan} \longrightarrow \mathrm{FinRep}(G)_{\mathbb{C}}$$

(objects are finite G-sets
morphisms are spans of finite G-sets

(objects are finite-dim reps of G
morphisms are intertwining operators

Morphisms in the image of F are "Hecke operators".

Thm: Given finite G-sets X & Y, every morphism from $F(X)$ to $F(Y)$ is a $\mathbb{C}$-lin. combination of morphisms of form $F(f)$ where $f : X \nrightarrow Y$.

Hecke operators

In $\mathrm{FinRep}(G)_{\mathbb{C}}$,

hom-sets are free $\mathbb{C}$-modules.

In $G\text{-}\mathrm{FinSpan}$,

hom-sets are free $\mathbb{N}$-modules.

$$F : G\text{-FinSpan} \longrightarrow \mathrm{FinRep}(G)_{\mathbb{C}}$$

We have

$$\hom(F(X), F(Y)) \cong \hom(X, Y) \otimes_{\mathbb{N}} \mathbb{C}$$

Thm.— For any field k, we have:

$$F : G\text{-FinSpan} \longrightarrow \mathrm{FinRep}(G)_k$$

& $$F : \hom(X, Y) \longrightarrow \hom(F(X), F(Y))$$

gives an isomorphism of k-vector spaces:

$$\hom(X, Y) \otimes_{\mathbb{N}} k \xrightarrow{\sim} \hom(F(X), F(Y)).$$

We were considering certain specific spans:

$$X : \binom{n}{k}_F \longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\;\; \binom{n+1}{k}_F$$

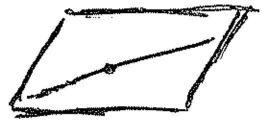
$$Y : \binom{n}{k}_F \longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\;\; \binom{n+1}{k+1}_F$$

for any field F.

$$V \subseteq F^n \mapsto V \subseteq F^n \qquad\qquad V \subseteq F^n \mapsto V \subseteq F^n \oplus F \subseteq F^{n+1}$$

$$\begin{tikzcd}
 & \binom{n}{k}_F \arrow[dl] \arrow[dr] & \\
\binom{n}{k}_F & & \binom{n+1}{k}_F
\end{tikzcd}$$

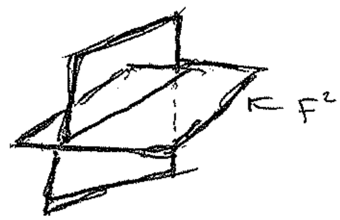
X:



$n=2$
$k=1$

$$\begin{tikzcd}
 & ? \arrow[dl] \arrow[dr] & \\
\binom{n}{k}_F & & \binom{n+1}{k+1}_F
\end{tikzcd}$$

Y:



$n=2$
$k=1$