We have been talking about the category of permutation representations of $G$ over the rig $\mathbb{N}$

$$\underline{\text{Perm } G\text{-Rep}}_{\underline{\mathbb{N}}}$$

$$\parallel$$

$$\underline{G\text{-Set}}_{\substack{\text{Hecke} \\ \text{Operators}}}$$

JB described a precursor to this, which can be decategorified to $\underline{\text{Perm } G\text{-Rep}}_{\underline{\mathbb{N}}}$

$$\underline{G\text{-Set}}_{\underline{\text{Span}}_{\text{Span map}}} \xrightarrow{\text{decategorification}} \underline{\text{Perm } G\text{-Rep}}_{\underline{\mathbb{N}}}$$

Last time JB gave a theorem, which we need to fix.

$$\underline{G\text{-Set}}_{\substack{\text{iso classes} \\ \text{of span}}} \longrightarrow \underline{G\text{-Set}}_{\substack{\text{Hecke} \\ \text{op}}} = \underline{\text{Perm } G\text{-Rep}}_{\underline{\mathbb{C}}} \hookrightarrow \underline{G\text{-Rep}}_{\underline{\mathbb{C}}}$$

$A, B$ G-Sets

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd} \qquad \begin{array}{c} \mathrm{Span}(A,B) \\ \shortparallel \\ \text{Free } \mathbb{N}\text{-module} \\ \text{on something} \end{array} \quad \overset{?}{\vdots} \quad \begin{array}{c} \mathrm{Hom}(R(A), R(B)) \\ \shortparallel \\ \text{Free vector space} \\ \text{over } \mathbb{C} \\ \text{on something} \end{array}$$

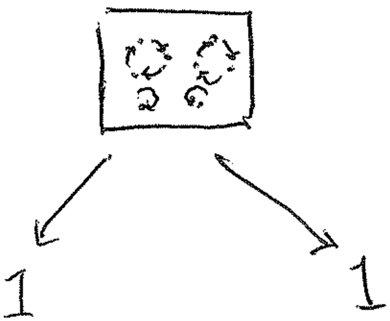
This was wrong!

Let's look at a counterexample.

Consider the terminal G-Set, $1$.

We were looking at "G-invariant spans".

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ 1 & & 1 \end{tikzcd}$$



Our decategorification looks like

$$\begin{tikzcd} \mathrm{Span}(A,B) \arrow[rr] \arrow[dr, "\text{``Take iso classes''}"'] & & V \arrow[dr] & \\ & \mathrm{Span}_{iso}(A,B) \arrow[ur, "\text{``tensor with } \mathbb{C}\text{''}"] \arrow[rr] & & \mathrm{Hom}(R(A), R(B)) \end{tikzcd}$$

# Categorification & Decategorification

Let's start with decategorification, which is a destructive simplification process. Thus, it is the easier of the two. Categorification requires some creativity.

$$\begin{tikzcd} \underline{\mathrm{CAT}} \arrow[r, "\text{``Take set of iso classes''}"'] & \underline{\mathrm{SET}} \end{tikzcd}$$

$$\begin{tikzcd} \underline{\mathrm{FINSET}} \arrow[r, "\text{``Cardinality''}"'] \arrow[d, "\text{``Free''}"'] & \underline{\mathbb{N}} \arrow[d, "="] \\ \underline{\mathrm{FD\ VSP}}_K \arrow[r, "\text{``Dimension''}"'] & \underline{\mathbb{N}} \end{tikzcd}$$

$$\begin{tikzcd} \underline{\mathrm{CAT}} \arrow[r, "\text{``Take set of components''}"'] & \underline{\mathrm{SET}} \end{tikzcd}$$

$$\begin{tikzcd} \underline{\mathrm{SPACE}} \arrow[r, "\pi_0"] \arrow[d, "\mathrm{id}"'] & \underline{\mathrm{SET}} \arrow[d, "\text{``}\exists\text{''}"] \\ \underline{\mathrm{SPACE}} \arrow[r, "\pi_{-1}", "\text{``Inhabitedness''}"'] & \text{TRUTH VALUES} \end{tikzcd}$$

turns a logical formula into a logical statement

Here is another example of decategorification:

$$\underline{\text{AB CAT}} \xrightarrow{\text{"Grothendieck Group"}} \underline{\text{AB GRP}}$$

$$\underline{\text{GPD}}_{\underline{\text{SPAN}}} \xrightarrow[\text{"}H_0\text{"}]{\text{"Degroupoidification"}} \underline{\text{VSP}}_{\underline{\mathbb{C}}}$$

little & littler fleas...

they bite!