"The [vector space] - enriched category

$$\underline{\text{Fin } G\text{-Set}}_{\text{Hecke } \subseteq\text{-op}}$$

is the decategorification of the category - enriched category

$$\underline{\text{Fin } G\text{-Set}}_{\underline{\text{Span}}}$$ "

We need to understand which type of decategorification we want to use & what enrichment means.

Enrichment of categories -

Category where the hom sets are not just sets but have some nice extra structure

We now want to see how enrichment & decategorification get along:

$$A \xrightarrow{\text{decat.}} B$$

$$\underline{A\text{-enriched cat}} \xrightarrow{\text{decat}} \underline{B\text{-enriched cat}}$$

$$C \longmapsto \boxed{\begin{array}{l}\text{take same objects} \\ \text{as in } C \text{, but} \\ \hom_{New}(X,Y) := \text{decat}(\hom_{old}(X,Y))\end{array}}$$

So we have built a new decategorification process out of an old one.

example:

$$\underline{\text{Set}} \xrightarrow{D} 1$$

$$\underline{\text{Set-enriched cat}} \longrightarrow \underline{1\text{-enriched cat}}$$

$\underline{1\text{-enriched cat}}$ has exactly 2 equivalence classes, which we can think of as corresponding to truth values.

Now we consider

truth-value-enriched cat

which gives us a partially ordered set.

$$\text{"Nice" } \underline{\text{Cat}} \xrightarrow{d} \underline{\underline{\mathbb{C}}\text{-vsp}}$$

$$\underline{\underline{\text{Cat}}\text{-enriched cat}} \xrightarrow{P(d)} \underline{[\underline{\mathbb{C}}\text{-vsp}]\text{-enriched cat}}$$

"Nice" means so nice that it's actually a groupoid in disguise, that is, it's the category of G-sets for some groupoid G.

Groupoid $\longmapsto$ Vector Space

Span $\longmapsto$ Linear Operator

by means of "transfer"

This is "Degroupoidification" or "H_0 for groupoids"