G-Set ⟼ Groupoid

$$G \times S \to S \quad \longmapsto \quad S/\!/G$$

$$(g,s) \mapsto gs \qquad\qquad s \xrightarrow{(g,s)} gs$$

Map from $G \times S \to S$ to $G \times S' \to S'$ ⟼ Functor $\Phi : S/\!/G \to S'/\!/G'$

$\phi : S \to S'$

$\varphi : G \to G'$

s.t.

$\phi(gs) = \varphi(g) \cdot \phi(s)$

$$\begin{array}{c} s \xrightarrow{(g,s)} gs \\ \big\downarrow \\ \phi(s) \xrightarrow{(\varphi(g),\,\phi(s))} \phi(gs) \end{array}$$

$S/G = \{[s]\}$ where $[gs] = [s]$.

$S/\!/G$ has $gs \xrightarrow[(g^{-1},\,s)]{\sim} s$

We get a
Category - enriched category
(= bicategory)
(= weak 2-category)

- groupoids $X$ as objects

- spans of groupoids
$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$
as morphisms

- equivalences of spans
$$\begin{tikzcd} S \arrow[r, "\sim"] \arrow[d] \arrow[drr] & S' \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$
as 2-morphisms

commuting up to natural isomorphism

Degroupoidification:

$D$ : [(finite) groupoids, spans, isos] $\longrightarrow$

[vector spaces, linear operators, equations]

If $X$ is a groupoid then what's the $0^{th}$ homology?

Given a category $C$, let

$$\underline{C} = \{\text{iso. classes of objects of } C\}$$

The $0^{th}$ homology of $X$ is the free vector space on $\underline{X}$:

$$H_0(X) = k[\underline{X}] = \left\{ \begin{array}{l} \text{formal finite } k\text{-linear} \\ \text{combinations of elements of } \underline{X} \end{array} \right\}$$

The $0^{th}$ cohomology of $X$ is

$$H^0(X,k) = H_0(X,k)^* = k^{\underline{X}}$$

$$= \{\text{functions } \psi : \underline{X} \to k\}$$

E.g.: $X = \mathrm{FinSet}_0$

$$H_0(X) \cong k[\underline{X}] \cong k[\mathbb{N}] \cong k[z] \qquad H^0(X) \cong k^{\underline{X}} \cong k^{\mathbb{N}} \cong k[[z]]$$

What about $X = (\mathrm{FinVect}_F)_0$ ?

The same!

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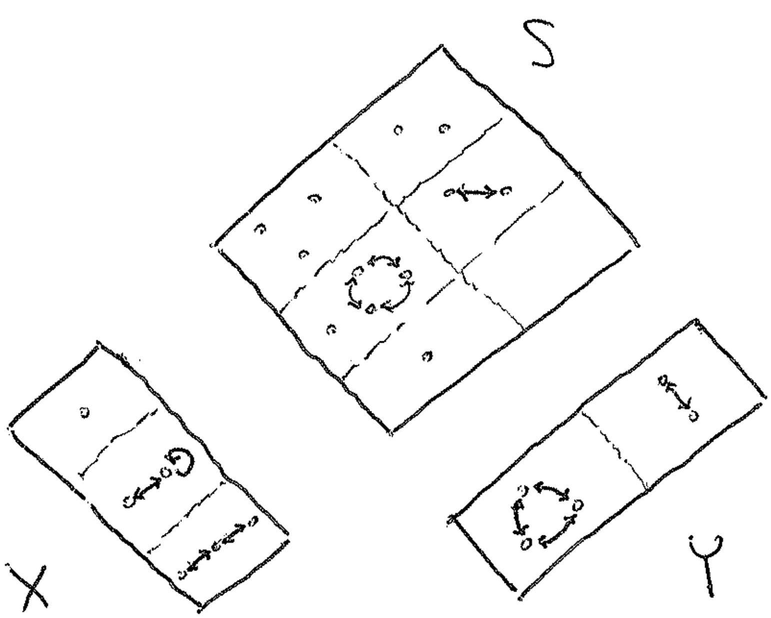
How does a span of finite groupoids

$$\begin{tikzcd} & S \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

give a linear operator $H_0(X) \longrightarrow H_0(Y)$



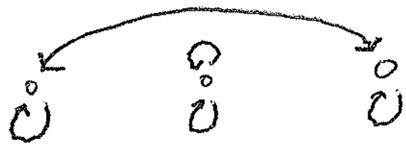

gives a $\underline{X} \times \underline{Y}$ - sized matrix of groupoids.

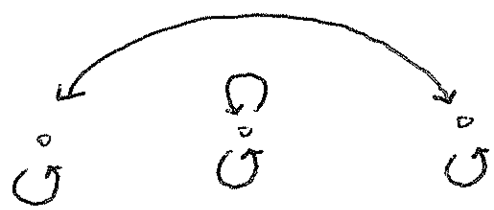
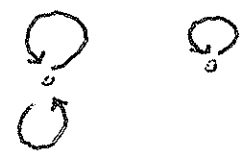
This becomes an $\underline{X} \times \underline{Y}$ - sized matrix of numbers

(in $k$ - a field of char 0!)

by turning each finite groupoid (in our matrix) into a number – it's <u>cardinality</u>!

$X =$  $\Longrightarrow \quad |X| = 3$

$X /\!/ \mathbb{Z}_2 =$  $\Longrightarrow \quad |X /\!/ \mathbb{Z}| = 3/2$

 $\simeq$ 

as a groupoid

$$|X /\!/ \mathbb{Z}_2| = |\mathbb{Z}_2| + |1|$$

$$= \frac{1}{2} + 1$$

So:

$$|X| = \sum_{[x] \in \underline{X}} \frac{1}{|\mathrm{Aut}(x)|}$$

Then check: $\quad |S /\!/ G| = |S| / |G|$