We are talking about groupoid cardinality again

$$\underline{\text{Tame Groupoid}} \xrightarrow{\text{"Groupoid Cardinality"}} \underline{\mathbb{R}^+}$$

This is a decategorification process given by

$$G \longmapsto \underset{x \text{ in } PI_0(G)}{\text{Sum}} \frac{1}{|PI_1(G,x)|}$$

Last time JB posed a puzzle to find the groupoid cardinality of the groupoid of finite sets.

$$\underline{\text{FinSet}}_{\text{bijections}} \longmapsto E$$

$$\underline{\text{2-Colored Fin Set}}_{\substack{\text{Color-}\\\text{preserving}\\\text{bijections}}} \longmapsto E^2$$

Here is another puzzle:

$$G = \underline{\text{FinMap}}_{\substack{\text{invert.} \\ \text{comm.} \\ \text{squares}}}$$

$$\begin{tikzcd} A \arrow[r, "\approx"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "\approx"'] & D \end{tikzcd}$$

An interesting open problem is to find an "interesting" groupoid whose cardinality is 2π.

We know want to piggyback on our decategorification process.

$$\underline{\text{Tame Groupoid}}_{\underline{\text{Tame Span}}} \xrightarrow{\text{"Degroupoidification"}} \underline{\mathbb{C}\text{-VSp}}_{\text{Lin Op}}$$

We will be using the process of groupoid cardinality at the level of $\underline{\text{Tame Span}}$ & Lin Op.

$$G \longmapsto H_0(G) = \underline{\mathbb{C}}\begin{bmatrix}\text{set of iso} \\ \text{classes of} \\ G\text{-objects}\end{bmatrix}$$

For now, we will take all groupoids to be finite.

$$\begin{tikzcd} G \\ S \arrow[u, "j"] \arrow[d, "k"'] \\ H \end{tikzcd} \longmapsto \begin{tikzcd} H_0(G) \arrow[dr, "j^!"] \arrow[dd] & \\ & H_0(S) \arrow[dl, "k_*"] \\ H_0(H) & \end{tikzcd}$$

$$H_0(G) \xrightarrow{j^!} H_0(S)$$

$$\underset{\text{(component of } G)}{Y} \longmapsto \frac{\displaystyle\sum_{\substack{X \\ \text{component} \\ \text{of } S \\ \text{in } j^{-1}(Y)}} |X| \cdot X}{|Y|}$$

$$\mathrm{Hom}_{\underline{\mathbb{C}}[G]}(\underline{\mathbb{C}}[X], \underline{\mathbb{C}}[Y])$$

X, Y finite G-sets

G finite group

$$(X \times Y)/\!/G \qquad H_0((X \times Y)/\!/G)$$

$$\| \qquad\qquad \|$$

"Groupoid of elements of the G-set $X \times Y$"

$\underline{\mathbb{C}}$ [iso classes in the gpd of elements of G-set $X \times Y$]

We want to show how spans get turned into linear operators. There is a useful map called "$\text{composition}_{X,Y,Z}$"

$$\mathrm{Hom}_{\underline{\mathbb{C}}[G]}(\underline{\mathbb{C}}[X], \underline{\mathbb{C}}[Y]) \otimes \mathrm{Hom}_{\underline{\mathbb{C}}[G]}(\underline{\mathbb{C}}[Y], \underline{\mathbb{C}}[Z])$$

$$\downarrow$$

$$\mathrm{Hom}_{\underline{\mathbb{C}}[G]}(\underline{\mathbb{C}}[X], \underline{\mathbb{C}}[Z])$$

We consider the span

$$
\begin{tikzcd}
(X \times Y)/\!/G \times (Y \times Z)/\!/G \\
(X \times Y \times Z)/\!/G \arrow[u] \arrow[d] \\
(X \times Z)/\!/G
\end{tikzcd}
$$