We've outlined this:

Theorem - There's a category

$$C = [\text{finite groupoids}, \text{ equivalence classes of spans of finite groupoids}]$$

& a functor

$$D : C \longrightarrow \mathrm{FinVect}_k$$

where k is any field of characteristic zero, ($\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, ...), given by:

$$\underset{\text{(finite groupoid)}}{X} \longmapsto \underset{\text{(0th homology)}}{H_0(X)}$$

$$\begin{tikzcd} & S \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd} \longmapsto \begin{tikzcd} & H_0(S) \arrow[dr, "q_*"] & \\ H_0(X) \arrow[ur, "p^!"] \arrow[rr] & & H_0(Y) \end{tikzcd}$$

We say 2 spans from X to Y are equivalent:

$$\begin{tikzcd}
S \arrow[rr, "\sim"] \arrow[d] \arrow[dr] & & S' \arrow[dll] \arrow[dl] \\
X & Y &
\end{tikzcd}$$

commuting up to a natural isomorphism.

(Equivalent spans give equal operators from $H_0(X)$ & $H_0(Y)$)

We compose spans of groupoids via "weak pullback".

$$\begin{tikzcd}
& & SS' \arrow[dl] \arrow[dr] \arrow[d, phantom, "\cong"] & & \\
& S \arrow[dl, "p"'] \arrow[dr, "q"] & \, & S' \arrow[dl, "p'"'] \arrow[dr, "q'"] & \\
X & & Y & & Z
\end{tikzcd}$$

An object of SS' is an object $s \in S$ & an object $s' \in S$ & an isomorphism $\alpha : q(s) \longrightarrow p'(s')$

A morphism in SS' is:

$$\begin{tikzcd} q(s_1) \arrow[r, "\alpha_1", "\sim"'] \arrow[d, "q(f)"'] & p'(s_1') \arrow[d, "p'(f')"] \\ q(s_2) \arrow[r, "\alpha_2", "\sim"'] & p'(s_2') \end{tikzcd}$$

where

$$f: s_1 \longrightarrow s_2$$
$$f': s_1' \longrightarrow s_2'.$$

This composition is associative & unital up to equivalence.

Using

$$D: C \longrightarrow \mathrm{FinVect}_k$$

we can turn a category A enriched over C into a category $\bar{D}(A)$ enriched over $\mathrm{FinVect}_k$.

$\bar{D}(A)$ has same objects as A.

& $$\hom_{\bar{D}(A)}(x,y) = D(\hom_A(x,y))$$

For us, $C = [\text{finite groupoids, equiv. classes of spans}]$

A category A enriched over C has:

- a class of objects $x, y, z, \ldots$
- for any objects x, y,

  a finite groupoid $\hom_A(x,y)$

- for any objects x, y, z,

$$\circ : \underbrace{\hom_A(x,y) \times \hom_A(y,z)}_{\text{product of groupoids}} \xrightarrow{\text{eq. class of spans}} \hom_A(x,z)$$

- for any object x,

$$\mathrm{id} : \underbrace{1}_{\substack{\text{terminal groupoid} \\ \text{(one object, one morphism groupoid)}}} \longrightarrow \hom_A(x,x)$$

equivalence class of spans

- associativity & l/r unit laws.

Example: if G is a finite group, there's a category enriched over $\mathbb{C}$ called $\mathrm{Hecke}(G)$:

- objects are finite G-sets $X, Y, \ldots$
- $\hom_{\mathrm{Hecke}(G)}(X,Y) = (X \times Y)/\!/G$
- composition, units ...

We call $(X \times Y)/\!/G$ the <u>Hecke groupoid</u> of X & Y.

$$\begin{tikzcd} (x,y) \arrow[d, "g"'] \\ (gx,gy) \end{tikzcd}$$

— a morphism in $(X \times Y)/\!/G$

Components of $(X \times Y)/\!/G$ are orbits in the G-set $X \times Y$.

For any G-set S,

$$\underline{S/\!/G} = S/G$$

so components of $(X \times Y)/\!/G$ form set $(X \times Y)/G$.

So:

$$\hom_{\bar{D}(\mathrm{Hecke}(G))}(X,Y) \simeq D(\hom_{\mathrm{Hecke}(G)}(X,Y))$$
$$= H_0(\hom_{\mathrm{Hecke}(G)}(X,Y))$$
$$\cong k^{(X\times Y)/G}$$

Elements of $X \times Y/G$ are also called:

"atomic invt. relations" between X & Y

a basis for

$$\hom_G(k^X, k^Y)$$

└ perm. rep of G on k^X

– "Hecke operators"!

So:

$$\hom_{\bar{D}(\mathrm{Hecke}(G))}(X,Y) \simeq D(X\times Y/\!/G)$$
$$= H_0(X\times Y/\!/G)$$
$$= k^{(X\times Y)/G}$$
$$\cong \hom_G(k^X, k^Y)$$

Fundamental Thm of Hecke Operators —

If G is a finite group & k is any field of char. 0,

$$\underline{D}(\mathrm{Hecke}(G)) \simeq \mathrm{FinPermRep}(G)_k$$

$\uparrow$ C-enriched cat $\qquad\qquad \uparrow$ $\mathrm{FinVect}_k$ - enriched cat

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Where next? Pascal's Triangle

$$\begin{tikzcd} & \mathrm{FinSet}_0 \arrow[dl, "1"'] \arrow[dr, "S \mapsto S+1"] & \\ \mathrm{FinSet}_0 & & \mathrm{FinSet}_0 \end{tikzcd}$$

$$\begin{tikzcd} & H_0(\mathrm{FinSet}_0) \arrow[dr, "p_*"] & \\ H_0(\mathrm{FinSet}_0) \arrow[ur, "1^!"] & & H_0(\mathrm{FinSet}_0) \end{tikzcd}$$

So this diagram is:

$$\begin{tikzcd} & k[x] \arrow[dr, "x^n \mapsto x^{n+1}"] & \\ k[x] \arrow[ur, "1"] \arrow[rr, "\text{``multiplication by } x\text{''}"'] & & k[x] \end{tikzcd}$$

$n \in \mathrm{FinSet}_0$

$[n] \in H_0(\mathrm{FinSet}_0)$

$\downarrow$

$x^n \in k[x]$

Next consider

$$Q = [\text{finite sets equipped w. a subset}]$$

$$= \sum_{n,k \geq 0} \begin{bmatrix} n \\ k \end{bmatrix}$$

Here we have 2 interesting spans:

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ Q & & Q \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dl] \arrow[dr] & \\ Q & & Q \end{tikzcd}$$

("add an element to the subset")

("add an element not in the subset")

These give 2 linear operators ~

"multiplication by x" & "multiplication by y"

$$(x+y)^n = \sum_{k \leq n} \binom{n}{k} x^{n-k} y^k$$

~ we can categorify this.

Similarly for the q-deformed version:

$$\mathrm{FinSet} \xrightarrow[K_q[x,y]]{} \mathrm{FinVect}_{\mathbb{F}_q}$$

Also a many-variable version involving multinomial coefficients.

"quantum n-space"

Categorify the quantum group $GL_q(n,k)$ – or upper triangular part!