

We've outlined this:

Theorem - There's a category

$$\mathcal{C} = [\text{finite groupoids, equivalence classes of spans of finite groupoids}]$$

† a functor

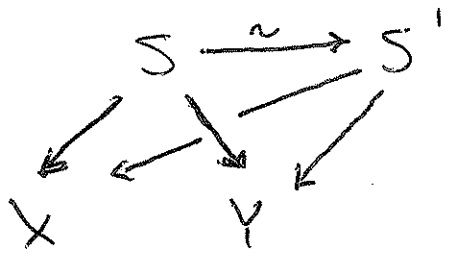
$$D: \mathcal{C} \longrightarrow \text{FinVect}_k$$

where  $k$  is any field of characteristic zero:  $(\mathbb{C}, \mathbb{R}, \mathbb{Q}, \dots)$ , given by:

$$\begin{array}{ccc} X & \longmapsto & H_0(X) \\ \text{(finite groupoid)} & & \text{(0th homology)} \end{array}$$

$$\begin{array}{ccc} & S & \\ \swarrow p & & \searrow q \\ X & & Y \end{array} \longmapsto \begin{array}{ccc} & H_0(S) & \\ \swarrow p_* & & \searrow q_* \\ H_0(X) & \longrightarrow & H_0(Y) \end{array}$$

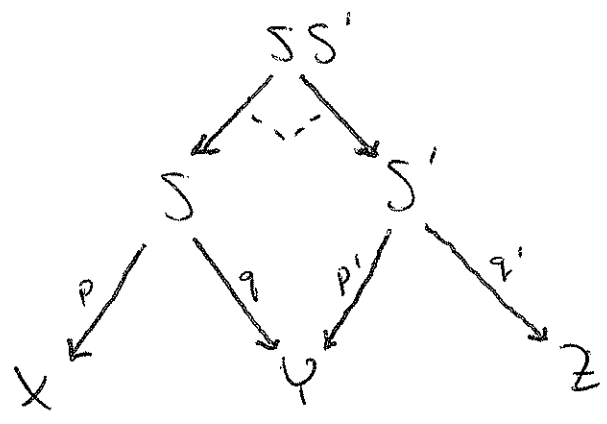
We say 2 spans from  $X$  to  $Y$  are equivalent:



commuting up to a natural isomorphism.

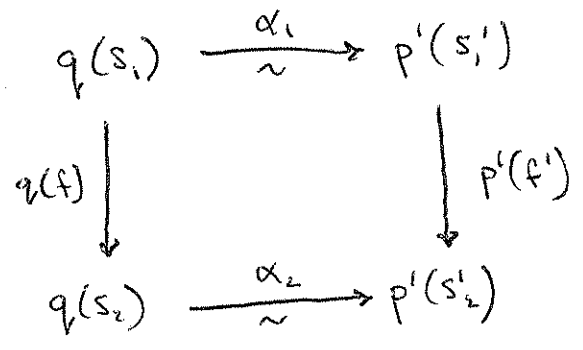
(Equivalent spans give equal operators from  $H_0(X) \otimes H_0(Y)$ )

We compose spans of groupoids via "weak pullback".



An object of  $SS'$  is an object  $s \in S$  ; an object  $s' \in S'$  ; an isomorphism  $\alpha : q(s) \rightarrow p'(s')$

A morphism in  $\mathcal{S}\mathcal{S}'$  is:



where

$$f: s_1 \longrightarrow s_2$$

$$f': s'_1 \longrightarrow s'_2$$

This composition is associative & unital up to equivalence.

Using

$$D: \mathcal{C} \longrightarrow \text{FinVect}_k$$

we can turn a category  $A$  enriched over  $\mathcal{C}$  into a category  $\bar{D}(A)$  enriched over  $\text{FinVect}_k$ .

$\bar{D}(A)$  has same objects as  $A$ .

$$\text{hom}_{\bar{D}(A)}(x, y) = D(\text{hom}_A(x, y))$$

For us,  $C = [\text{finite groupoids, equiv. classes of spans}]$

A category  $A$  enriched over  $C$  has:

- a class of objects  $x, y, z, \dots$
- for any objects  $x, y$ ,  
a finite groupoid  $\text{hom}_A(x, y)$

- for any objects  $x, y, z$ ,

$$o : \text{hom}_A(x, y) \times \text{hom}_A(y, z) \longrightarrow \text{hom}_A(x, z)$$

(product of groupoids)
(eq. class of spans)

- for any object  $x$ ,

$$\text{id} : I \longrightarrow \text{hom}_A(x, x)$$

(terminal groupoid  
 (one object, one morphism groupoid))

equivalence class of spans

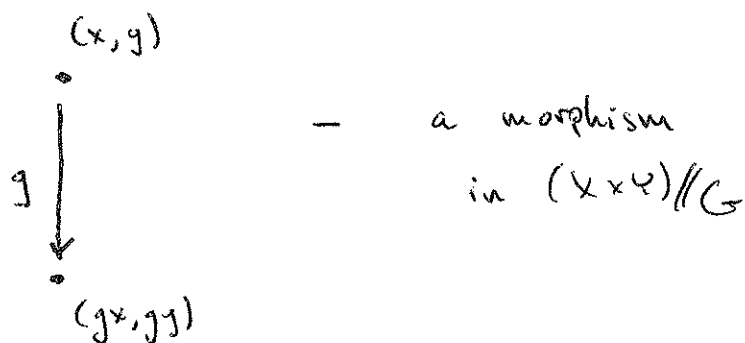
- associativity  $\hat{=}$  l/r unit laws.

(5)

Example: if  $G$  is a finite group, there's a category enriched over  $\mathbb{C}$  called  $\text{Hecke}(G)$ :

- objects are finite  $G$ -sets  $X, Y, \dots$
- $\text{hom}_{\text{Hecke}(G)}(X, Y) = (X \times Y) // G$
- composition, units  $\dots$

We call  $(X \times Y) // G$  the Hecke groupoid of  $X \ni Y$ .



Components of  $(X \times Y) // G$  are orbits in the  $G$ -set  $X \times Y$ .

For any  $G$ -set  $S$ ,

$$\underline{S // G} = S / G$$

so components of  $(X \times Y) // G$  form set  $(X \times Y) / G$ .

S<sub>0</sub>:

$$\begin{aligned}
\text{hom}_{\overline{D}(\text{Hecke}(G))}(X, Y) &= D(\text{hom}_{\text{Hecke}(G)}(X, Y)) \\
&= H_0(\text{hom}_{\text{Hecke}(G)}(X, Y)) \\
&\cong K^{(X \times Y)/G}
\end{aligned}$$

Elements of  $X \times Y/G$  are also called:

"atomic invt. relations" between  $X \hat{=} Y$

a basis for

$$\begin{aligned}
&\text{hom}_G(K^X, K^Y) \\
&\quad \downarrow \text{perm. rep of } G \\
&\quad \text{on } K^X
\end{aligned}$$

- "Hecke operators"!

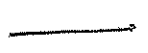
S<sub>0</sub>:

$$\begin{aligned}
\text{hom}_{\overline{D}(\text{Hecke}(G))}(X, Y) &= D(X \times Y // G) \\
&= H_0(X \times Y // G) \\
&= K^{(X \times Y)/G} \\
&\cong \text{hom}_G(K^X, K^Y)
\end{aligned}$$

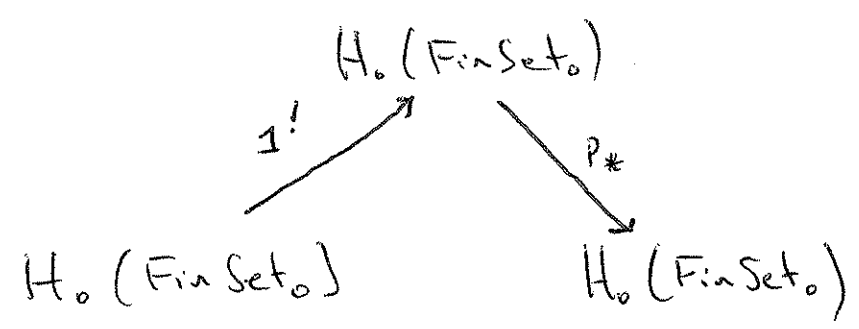
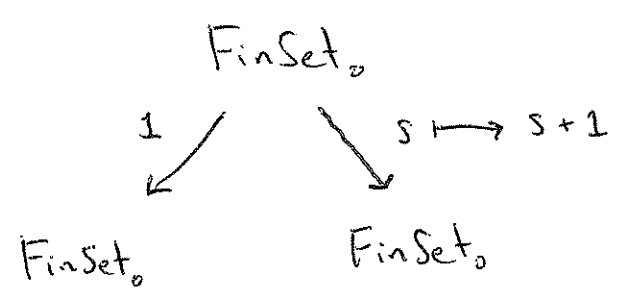
# Fundamental Thm of Hecke Operators —

If  $G$  is a finite group &  $k$  is any field of char. 0,

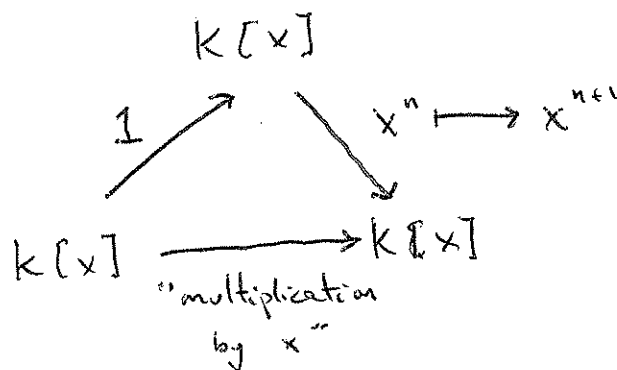
$$\begin{array}{ccc} \overline{D}(\text{Hecke}(G)) & \cong & \text{Fin Perm Rep}(G)_k \\ \uparrow & & \uparrow \\ \mathbb{C}\text{-enriched cat} & & \text{FinVect}_k\text{-enriched cat} \end{array}$$



Where next?      Pascal's Triangle



So this diagram is :



$$n \in \text{FinSet}_0$$

$$[n] \in H_0(\text{FinSet}_0)$$



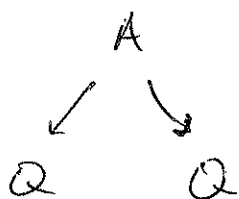
$$x^n \in K[x]$$

Next consider

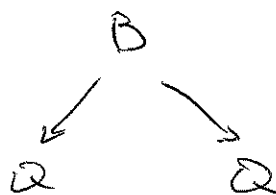
$$Q = [\text{finite sets equipped w. a subset}]$$

$$= \sum_{n, k \geq 0} \binom{n}{k}$$

Here we have 2 interesting spans :



"add an element to the subset"



"add an element not in the subset"



These give 2 linear operators ~

"multiplication by x" ; "multiplication by y"

$$(x+y)^n = \sum_{k \leq n} \binom{n}{k} x^{n-k} y^k$$

~ we can categorify this,

Similarly for the q-deformed version:

$$\text{FinSet} \xrightarrow{K_q[x,y]} \text{FinVect}_{F_q}$$

Also a many-variable version involving multinomial coefficients.

"quantum n-space"

Categorify the quantum group  $GL_q(n, k)$  -  
or upper triangular part!