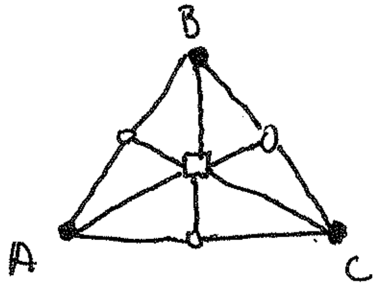
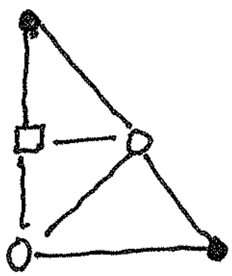
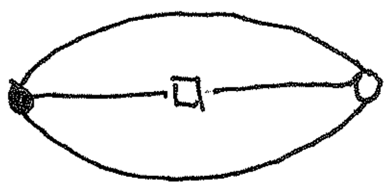
Pictures of "orbi-simplex" of $G \subseteq 3!$



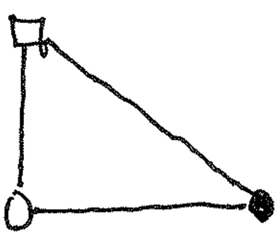
G trivial



$G = \{(1), (12)\}$

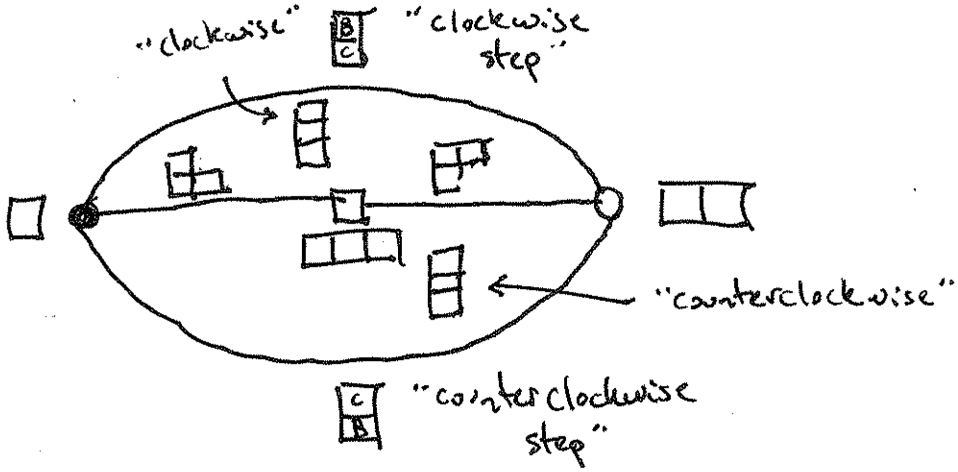


$G = \{(1), (123), (132)\}$



$G = 3!$

Let's look at Young diagram labels for the rotation group picture



Each piece in the orbi-simplex of a transformation group

$$G \subseteq S!$$

represents a G-orbit of flags on S of the type corresponding to the Young diagram label.

The structure preserved by the rotation group is "orientation".

We can associate to this structure an axiomatic theory.

We have a binary predicate "CW"

$CW(x, y)$ "Clockwise step from x to y"

$CCW(x, y)$ "counterclockwise step from x to y"

=

Every vertical Young diagram becomes an n-ary predicate in the theory.

An example of an axiom from the picture

$$\forall x, y \ (CW(x,y) \vee CCW(x,y) \vee (x = y))$$

What do these pictures & the corresponding axiomatic representations have to do with group representation theory?

Let's see how transformation groups are related to group representations.

A transformation group

$$G \subseteq S!$$

gives a group action on the set S.

We would like to obtain from this a group representation on a vector space.

$$\begin{tikzcd} G \subseteq S! \arrow[d, hook, "\text{“inclusion of permutation matrices”}"] \\ G \subseteq \mathrm{Mat}_S(\underline{\mathbb{C}}) \end{tikzcd}$$

$$G \longrightarrow \underline{\mathrm{Set}} \xrightarrow{\text{“Free”}} \underline{\mathrm{VSP}}_{\underline{\mathbb{C}}}$$

this composite functor is essentially the same as the inclusion of permutation matrices.

<u>Theorem</u>: Let $G \subseteq S!$ be a finitary transformation group. Let R be the complex representation of G obtained by:

$$\begin{tikzcd} G \arrow[r, "\text{“Action of } G \text{ on } S\text{”}"] \arrow[rr, bend right, "R"'] & \underline{\mathrm{Set}} \arrow[r, "\text{“Free”}"] & \underline{\mathrm{VSP}}_{\underline{\mathbb{C}}} \end{tikzcd}$$

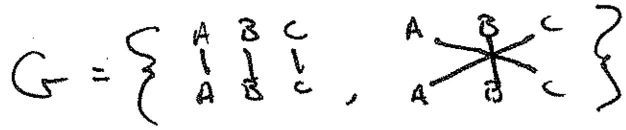
Then the hom-space

$$\mathrm{Hom}_G(R, R)$$

is a complex vector space with basis given by the orbits of G acting on S^2.
(i.e. Young diagram labels of the form)

Let's look at an example with G, the two element ~~subgroup~~ subgroup of S!

$S = \{A, B, C\}$



$R = \langle A, B, C \rangle$

$$\begin{array}{c|ccc} & A & B & C \\ \hline A & & & 1 \\ B & & 1 & \\ C & 1 & & \end{array}$$

$$\begin{bmatrix} D & E & F \\ G & H & J \\ K & L & M \end{bmatrix} \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} = \begin{bmatrix} F & E & D \\ J & H & G \\ M & L & K \end{bmatrix}$$

$$\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \begin{bmatrix} D & E & F \\ G & H & J \\ K & L & M \end{bmatrix} = \begin{bmatrix} K & L & M \\ G & H & J \\ D & E & F \end{bmatrix}$$

So we have a 5-dim v.s. w. basis

$$\begin{bmatrix} 1 & & \\ & & \\ & & 1 \end{bmatrix} \quad \begin{bmatrix} & & 1 \\ & & \\ 1 & & \end{bmatrix} \quad \begin{bmatrix} & 1 & \\ & & \\ & 1 & \end{bmatrix} \quad \begin{bmatrix} & & \\ 1 & & 1 \\ & & \end{bmatrix} \quad \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix}$$

We want to interpret how these orbits can be viewed as relationships between things in our axiomatic theory.