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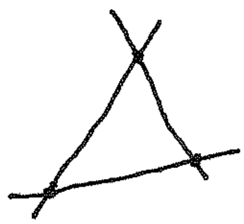
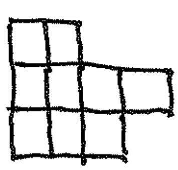
Set Theory	Projective Geometry
Finite sets	Finite-dim vector spaces
n	F^n or FP^{n-1}
	1d subspaces — points
$n = 3$ 	 FP^2
Here a "line" is just 2 points	
points, pairs, triples, ...	points, lines, planes, ...

A D-flag in n is:

$$\emptyset = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_k = n$$

where

$D =$  $n_1, n_2, \ldots, n_k$ $\quad n_1 + \cdots + n_k = n$

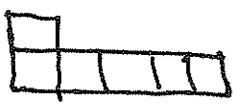
$$|X_i \setminus X_{i-1}| = n_i .$$

A D-flag in F^n is subspaces:

$$\{0\} = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_k = F^n$$

where D is as before - an uncombed n-box Young diagram & $\dim(X_i/X_{i-1}) = n_i$.

Let's <u>count</u> the D-flags in n & F^n where $F = F_q$.

$D =$  $n_1 = 1$, $n_2 = n-1$

$|D(n)| = n$

(D-flags on n = points of n

$|D(F^n)| = [n]_q$

$D(F^n)$ = points of FP^{n-1}

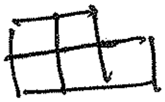
$$:= \frac{q^n - 1}{q-1}, \quad q\text{-integer}$$

categorification

$$FP^{n-1} = \frac{F^n - \{0\}}{F - \{0\}} \qquad |FP^{n-1}| = \frac{q^n - 1}{q-1}$$

Decategorification

Next:

$D =$  $n_1 = 2$, $n_2 = 3$ $\qquad n = 5$

$D(5)$ = 2-elt subsets of 5

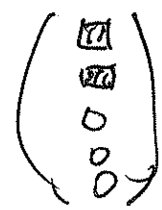
$$|D(5)| = \binom{5}{2} = \frac{5!}{2! \times 3!}$$

$2! \times 3!$ is the Young subgroup of permutations of 5 preserving the D-flag.

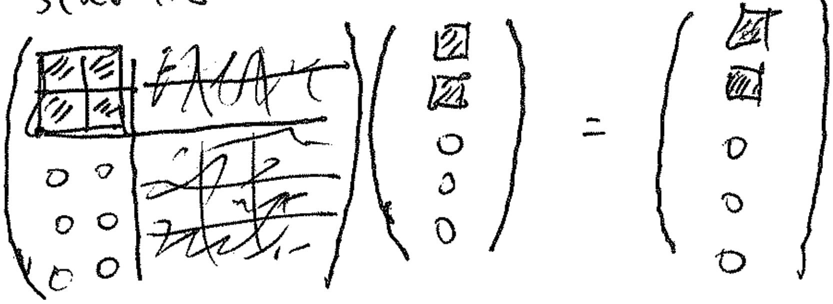
How about D-flags on F^5?

$$D(F^5) = \{ 2d \text{ subspaces of } F^5 \}$$

$$\cong \{ \text{line in } FP^4 \}$$

$GL(5, F)$ acts transitively on $D(F^5)$; 

the stabilizer subgroup of a D-flag



This group of matrices is called a <u>parabolic</u> subgroup of $GL(n)$ - preserves a D-flag.

$$|D(F^5)| = \frac{|GL(5,F)|}{\left|\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}\right|}$$

$$= \frac{|GL(5,F)|}{|(GL(2)\times GL(3)) \ltimes F^{2\times 3}|}$$

So, what's

$$|GL(n, F_q)| \;?$$

$$|GL(n,F_q)| = |(F_q^n - \{0\}) \times (F_q^n - F_q) \times (F_q^n - F_q^2) \times \cdots (F_q^n - F_q^{n-1})|$$

$$= (q^n-1)(q^n-q)\cdots(q^n-q^{n-1})$$

$$= \frac{q^n-1}{q-1}\cdot q \cdot \frac{q^{n-1}-1}{q-1} \cdots q^{n-1}\frac{q-1}{q-1}\cdot (q-1)^n$$

$$= [n]_q!\,(q-1)^n q^{\binom{n}{2}}$$

$$|D(F^5)| = \frac{|GL(5,F)|}{|GL(2)|\,|GL(3)|\,|F^{2+3}|}$$

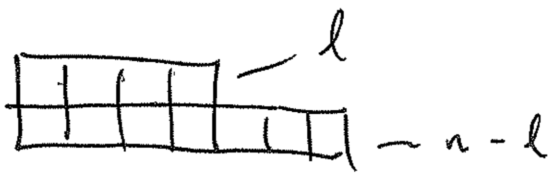
$$= \frac{[5]_q!\,(q-1)^5 q^{\binom{5}{2}}}{[2]_q!\,(q-1)^2 q^{\binom{2}{2}}[3]_q!\,(q-1)^3 q^{\binom{3}{2}}\cdot q^6}$$

$$= \frac{[5]_q!}{[2]_q!\,[3]_q!} \cdot \frac{q^{\binom{5}{2}}}{q^{\binom{2}{2}}\, q^{\binom{3}{2}}\, q^{6}}$$

$$= \binom{5}{2}_q$$

q-binomial coefficient

If

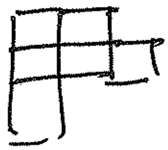
$D =$ 

$D(F^n)$ = Grassmannian of ℓ-dim subspaces of F^n

$= Gr(n, \ell)$

$$= \binom{n}{\ell}_F$$

Now:

$D =$  $n_1 = 2$, $n_2 = 3$, $n_3 = 1$ $\qquad n = 2+3+1 = 6$

$$D(6) \cong \binom{6}{2} \times \binom{4}{3}$$

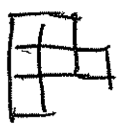
$$\cong \frac{6!}{2! \times 4!} \times \frac{4!}{3! \times 1!}$$

$$\cong \frac{6!}{2! \times 3! \times 1!} \longleftarrow \text{Young subgroup preserving D-flag.}$$

In general

$$D(n) = \frac{n!}{n_1! \times \cdots \times n_k!} \quad \text{a multinomial coefficient}$$

$$= \binom{n}{n_1, n_2, \ldots, n_k}$$

Finally: $D =$ 

$$D(F^6) = \binom{6}{2}_F \text{"}\times\text{"} \binom{4}{3}_F$$

$$|D(F^6)| = \binom{6}{2}_q \binom{4}{3}_q = \frac{[6]_q!}{[2]_q!\,[3]_q!\,[1]_q!} = \binom{6}{2,3,1}_q$$

For any n-box uncombed Young diagram D:

$$D(F^n) = \binom{n}{n_1, n_2, \ldots, n_k}_F$$

$$|D(F^n)| = \frac{[n]_q!}{[n_1]_q! \cdots [n_k]_q!}$$

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{n-1}$$

$$FP^{n-1} = [n]_F \cong \frac{F^n - \{0\}}{F - \{0\}} \cong 1 + F + F^2 + \cdots + F^{n-1}$$