Lecture 6 John Baez

We left off fixing an uncombed Young diagram D with n boxes & we learned how to count D-flags on an q-element set

$$\begin{tikzcd}
{|D(n)| = \binom{n}{n_1,\dots,n_k} \in \mathbb{N}} & & {|D(\mathbb{F}_q^n)| = \binom{n}{n_1,\dots,n_k}_q} \arrow[ll, "\lim_{q\to 1}"'] \arrow[dr, phantom, sloped, "\in"] & \\
 & & & \hat{\mathbb{Z}}(q) \\
{D(n) \in \mathrm{FinSet}} \arrow[uu, "\text{decategorification}\ 1\cdot 1"] & & {D(\mathbb{F}_q^n) \in ??} \arrow[ll, dashed, "``\lim_{q\to 1}\text{''}\ ?"'] \arrow[uu, "1\cdot 1"'] & \\
 & & \text{Motives?} &
\end{tikzcd}$$

Thm: Clearly

$$\binom{n}{n_1,\dots,n_k}_q = \frac{[n]_q!}{[n_1]_q!\cdots[n_k]_q!} \in \mathbb{Z}(q)$$

but in fact

$$\binom{n}{n_1,\dots,n_k}_q \in \mathbb{N}[q]\ !$$

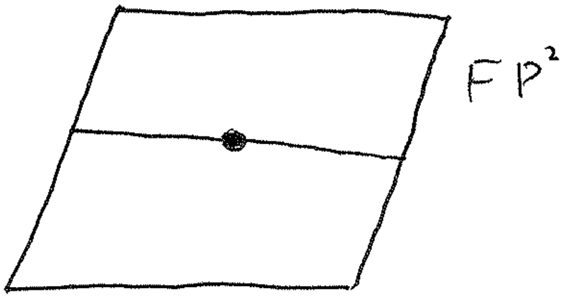
Easy example: $D =$  $n = 3$

$$D(\mathbb{F}_q^3) = \mathbb{F}_q P^2 = \frac{\mathbb{F}_q^3 - \{0\}}{\mathbb{F}_q - \{0\}}$$

$$|D(\mathbb{F}_q^3)| = \frac{q^3-1}{q-1} = 1 + q + q^2$$

since

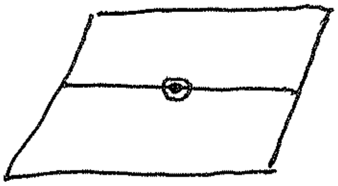
$$\mathbb{F}_q P^2 = 1 + \mathbb{F}_q + \mathbb{F}_q^2$$



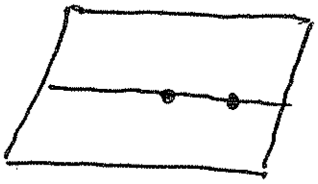
Pick a total flag –

D_0-flag w. $D_0 =$ 

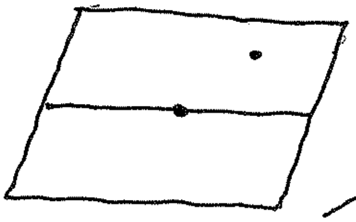
A D-flag could relate to this via:



1 way to be the point



$|F|$ ways to lie on the line



$|F^2|$ ways to lie on the plane

So, $$FP^2 \cong \overbrace{1 + F + F^2}^{\text{Bruhat classes}}$$

A flag variety $D(F^n)$ is the disjoint union of *Brohat classes*, obtained by choosing a *total flag* - a D_0-flag with

$$D_0 =$$

.

Write

$$D(F^n) \times D_0(F^n) = \coprod_i B_i$$

where B_i are $GL(n)$-orbits. Next choose

$$\Phi \in D_0(F^n)$$

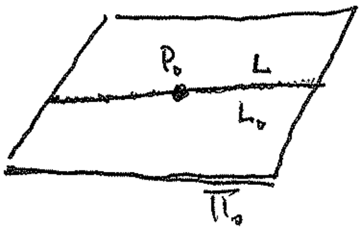
& write

$$D(F^n) = \coprod_i \{\phi : (\phi, \Phi) \in B_i\}$$

↑

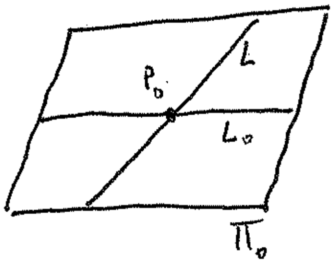
& these are *Brohat classes*.

$n = 4$ $D_0 =$  $D =$ 

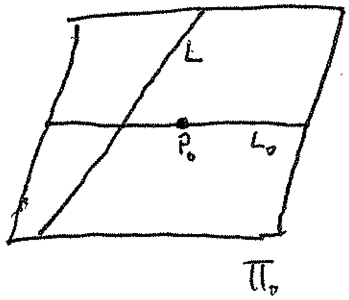


$\Phi = (P_0, L_0, \Pi_0)$

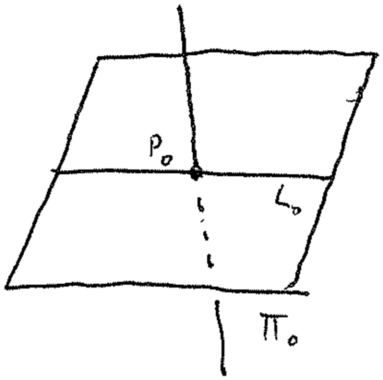
$L = L_0$ this Bruhat class has 1 element



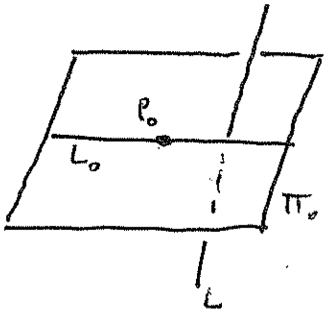
$P_0 \subseteq L \subseteq \Pi_0$ (and no better) F



$L \subseteq \Pi_0$ (and no better) F^2

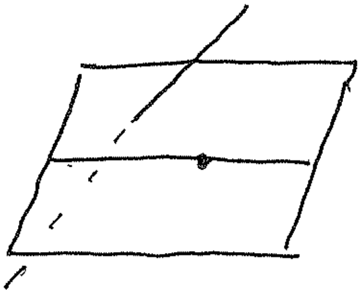


$P_0 \subseteq L$ (and no better) F^2



$\exists\ P' \quad P' \leq L_0$ & $P' \leq L \qquad F^3$

(and no better)



& no better $\qquad F^4$

$$D(F^4) \cong 1 + F + F^2 + F^3 + F^4$$

$$|D(F_q^4)| = \binom{4}{2}_q = 1 + q + 2q^2 + q^3 + q^4$$

This is not obvious from the definition:

$$\binom{4}{2}_q = \frac{[4]_q!}{[2]_q!\,[2]_q!}$$

$$= \frac{1\cdot(q+1)\cdot(q^2+q+1)\cdot(q^3+q^2+q+1)}{1\cdot(q+1)\ \cdot\ 1\cdot(q+1)}$$

$$\frac{1\cdot(q+1)(q^2+q+1)(q^3+q^2+q+1)}{(q+1)\,(q+1)} \overset{\text{in base } q}{=} \frac{1\cdot 11\cdot 111\cdot 1111}{11\cdot 11}$$

$$= \frac{111\cdot 1111}{11}$$

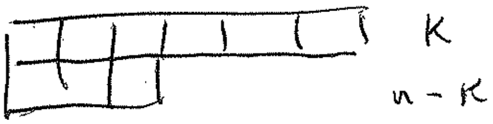
$$= \frac{123321}{11}$$

$$= 11211$$

$$= q^4 + q^3 + 2q^2 + q + 1$$

If it's in $\mathbb{N}[q]$,
it's a palindromic polynomial.

Let's show $|D(F^n)| \in \mathbb{N}[q]$ when D has 2 rows :



$D(F^n) = \binom{n}{K}_F$ is a Grassmannian of K-planes in F^n

Then we see the theorem in general!

Keeping in mind our example from earlier, we choose a basis for each cell.

$P_0 = \langle (1,0,0,0) \rangle$

$L_0 = \langle (1,0,0,0), (0,1,0,0) \rangle$

$\Pi_0 = \langle (1,0,0,0), \dots \rangle$

$L = L_0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	1
$P_0 \subseteq L \subseteq \Pi_0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{pmatrix}$	F
$L \subseteq \Pi_0$	$\begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \end{pmatrix}$	F^2
$P_0 \subseteq L$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{pmatrix}$	F^2
$\exists P' \quad P' \subseteq L_0$ & $P' \subseteq L$	$\begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{pmatrix}$	F^3
& no better	$\begin{pmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix}$	F^4