Lecture 8 John Baez

The q-deformed Pascal's Triangle

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

k = 0

n = 0 $\qquad$ 1

k = 1

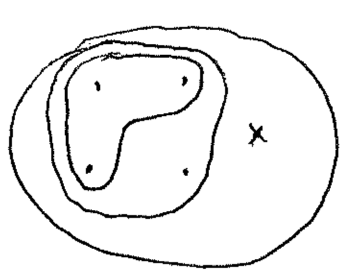
n = 1 $\qquad$ 1 $\quad$ 1

n = 2 $\qquad$ 1 $\quad$ 2 $\quad$ 1

1 $\quad$ 3 $\quad$ 3 $\quad$ 1

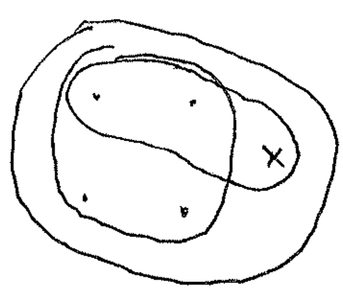
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$\left\{\begin{matrix}k\text{-elt subsets}\\ \text{of } n\end{matrix}\right\} \quad \left\{\begin{matrix}k\text{-elt subsets}\\ \text{of } n-1\end{matrix}\right\} + \left\{\begin{matrix}(k-1)\text{-elt subsets}\\ \text{of } n-1\end{matrix}\right\}$$



$\binom{4}{3}$

not including special point



$\binom{4}{2}$

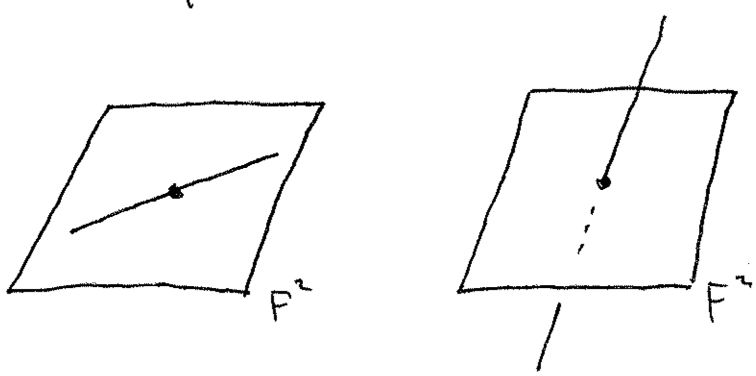
including special point

Let's look at the q-deformed case.

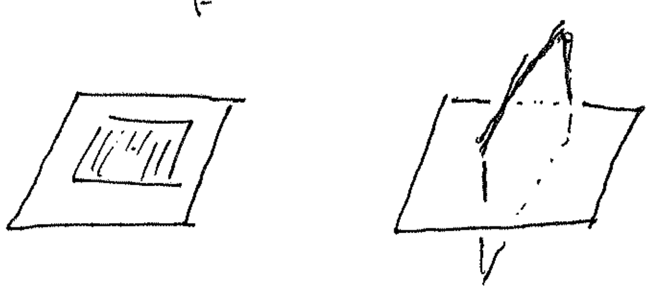
$$\underbrace{\binom{n}{k}_q}_{\left\{\begin{smallmatrix}k\text{-dim subspaces}\\ \text{of } \mathbb{F}_q^n\end{smallmatrix}\right\}} = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q$$

Example:

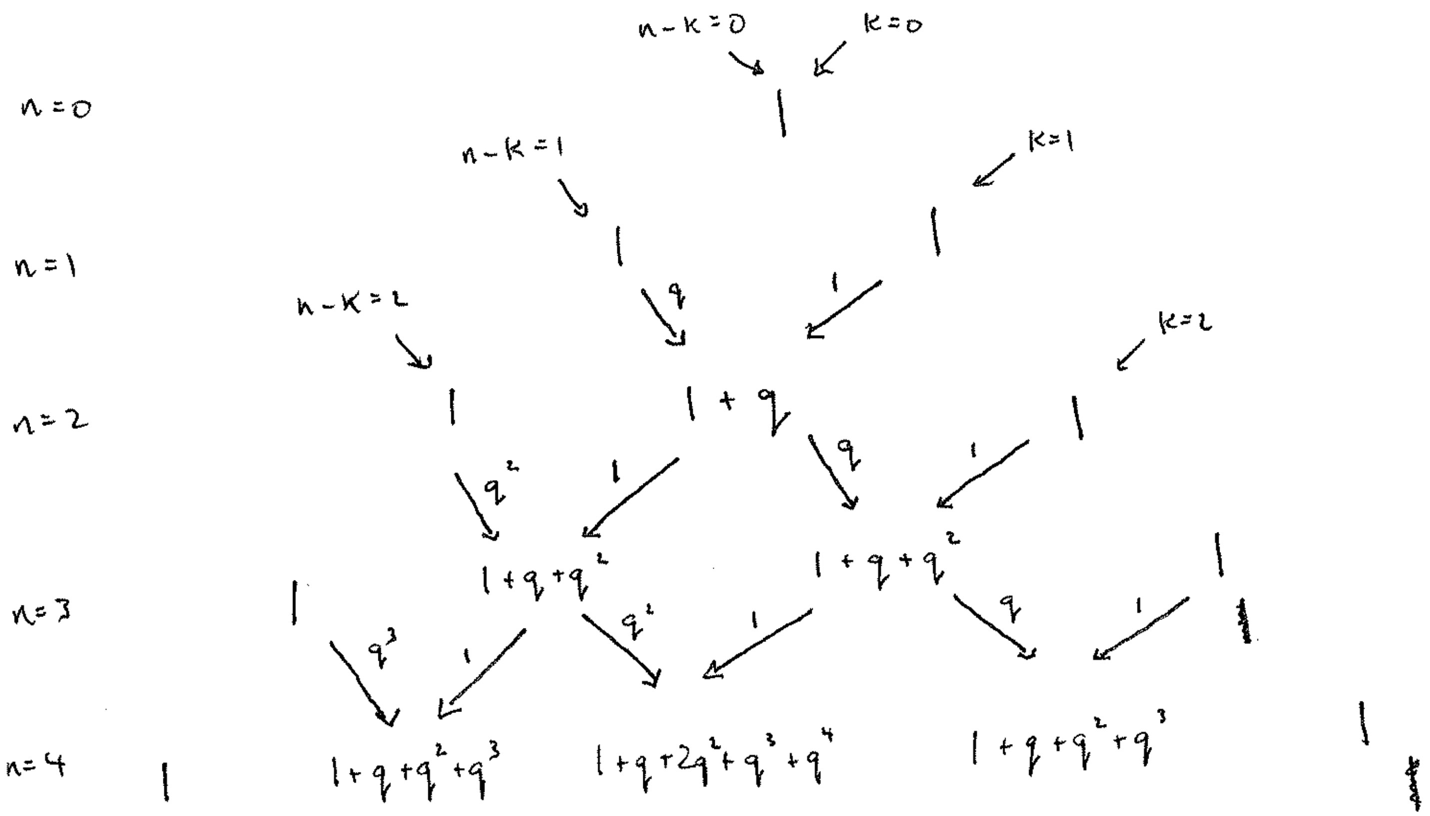
$$\binom{3}{1}_F \cong \binom{2}{1}_F + F^2 \times \binom{2}{0}_F$$



$$\binom{3}{2}_F \cong \binom{2}{2}_F + F^1 \times \binom{2}{1}_F$$

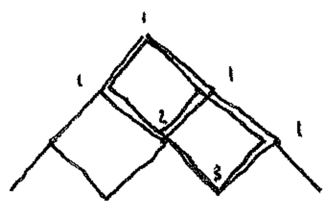


$$\binom{n}{k}_F \cong \binom{n-1}{k}_F + F^{n-k}\binom{n-1}{k-1}_F$$

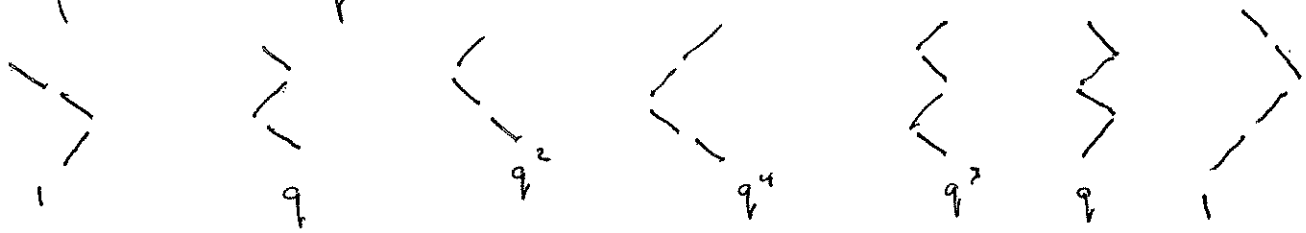


This is the q-deformed Pascal's triangle.

The entries in the regular Pascal's triangle count the number of paths from the top.

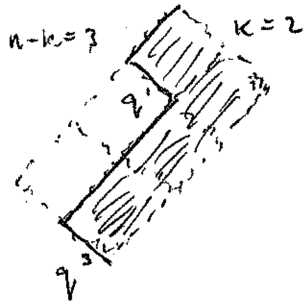


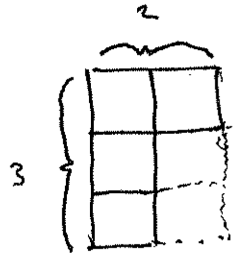
In the q-deformed case we are counting paths weighted by a power of q.



which we can see by counting the number of boxes enclosed on the left by our path.

$\binom{5}{2}$





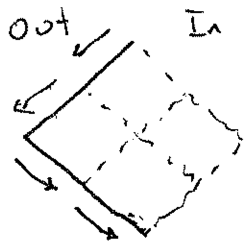
$$\binom{n}{k} = \left| \left\{ \begin{array}{l} \text{combed Young diagrams} \\ \text{w. } \le k \text{ columns } ; \\ \quad \le n-k \text{ rows} \end{array} \right\} \right|$$

$$\binom{n}{k}_q = \sum_{\substack{\text{such Young} \\ \text{Diagrams } D}} q^{\#\text{ of boxes of } D}$$

$$\binom{4}{2}_q = 1 + q + 2q^2 + q^3 + q^4$$

for q^4

$$\begin{pmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix}$$



$\swarrow$ $(1\ 0\ 0\ 0)$

$\swarrow$ $(0\ 1\ 0\ 0)$

$\searrow$ $(0\ 0\ 1\ 0)$

$\searrow$ $(0\ 0\ 0\ 1)$

for q^2

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{pmatrix}$$