Lie Theory Through Examples

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Lecture 1

1 Introduction

In this class we'll talk about a classic subject: the theory of simple Lie groups and simple Lie algebras. This theory ties together some of the most beautiful, symmetrical structures in mathematics: Platonic solids and their higher-dimensional cousins, finite groups generated by reflections, lattice packings of spheres, incidence geometries, symmetric spaces, and more. We shall explore this web of ideas through *examples*, starting with easy 'classical' ones and working up to 'exceptional' ones such as the 248-dimensional Lie group E_8 — which has recently been in the news a lot.

So: we'll state a bunch of general theorems, but not prove most of them. Instead, we'll see how they work in examples. This is meant as a corrective to the usual textbook treatments, which are heavy on proofs but light on examples.

So: if you want to see the proofs of theorems I state, read some books! There are a lot of books on this subject, so I'll just mention four of my favorites, in rough order of increasing sophistication:

- Brian Hall, *Lie Groups, Lie Algebras, and Representations*, Springer Verlag, Berlin, 2003. (A great first place to start.)
- William Fulton and Joe Harris, *Representation Theory - a First Course*, Springer Verlag, Berlin, 1991. (A more advanced but still friendly introduction to finite groups, Lie groups, Lie algebras and their representations, including the classification of simple Lie algebras. One great thing is that this book has lots of *pictures* of root systems, and works slowly up a ladder of examples of these before blasting the reader with abstract generalities.)
- J. Frank Adams, *Lectures on Lie Groups*, University of Chicago Press, Chicago, 2004. (A very elegant introduction to the theory of semisimple Lie groups and their representations, without the morass of notation that tends to plague this subject. But it's a bit terse, so you may need to look at other books to see what's really going on in here!)
- Daniel Bump, *Lie Groups*, Springer Verlag, Berlin, 2004. (A great tour of the vast and fascinating panorama of mathematics surrounding groups, starting from really basic stuff and working on up to advanced topics. The nice thing is that it explains stuff without feeling the need to prove every statement, so it can cover more territory.)

We'll be concerned with 'Dynkin diagrams', which are certain bunches of dots connected by arrows, sometimes with extra decorations on them. Dynkin diagrams are great because each Dynkin diagram D describes a bunch of different things, that are all related. For example, it describes:

- A **simply-connected complex simple Lie group** G. I'll assume you know what a Lie group is: a smooth manifold with smooth product and inverse operations making it into a group. A **complex manifold** is one that's been covered by coordinate charts that look like $\mathbb{C}^n$ for some n, with *complex-analytic* transition functions. We can define complex-analytic functions between complex manifolds. In a **complex Lie group**, the multiplication and inverse operations are required to be complex-analytic.

  We say a Lie group is **simple** if it has no normal subgroups except discrete subgroups. This is like the usual definition of 'simple group', but with a little extra slack thrown in so we don't worry too much about discrete normal subgroups. The reason is that if G is a Lie group and

N is a normal subgroup, G/N will be a Lie group whenever N is closed as a subspace of G — but the Lie algebra of G/N will be isomorphic to that of G precisely when N is discrete. If G is any Lie group, its universal cover $\tilde{G}$ will be simply connected, and we have $G \cong \tilde{G}/N$ for some discrete normal subgroup. So, for any simple Lie group, we can always find a *simply connected* simple Lie group with the same Lie algebra. We do this to, umm, simplify things.

- A **complex simple Lie algebra** $\mathfrak{g}$, namely the Lie algebra of G. I'll assume you know what a Lie algebra is: a vector space with a bilinear bracket operation that's antisymmetric and satisfies the Jacobi identity. A **complex** Lie algebra is one where we use a complex vector space. It's **simple** if it has no ideals (except 0 and all of $\mathfrak{g}$), so we can't take a quotient and get a smaller Lie algebra. It turns out that a Lie group is simple (as defined above) if and only if its Lie algebra is simple.

- A **compact simply-connected simple Lie group** K, which is a maximal compact subgroup of G. A **maximal compact subgroup** of a Lie group is a subgroup that's compact and not contained in any larger compact subgroup. G will have a bunch of maximal compact subgroups, but they'll all be isomorphic — in fact, they're all conjugate to each other. So, people often talk about 'the' maximal compact subgroup.

  In algebra, complex numbers are easier to work with than real numbers. In analysis, compact spaces are very nice. But a complex simple Lie group can never be compact. So, we have a choice: work with G and take advantage of the fact that it's complex, or work with K and take advantage of the fact that it's compact. We can jump back and forth and do whatever is convenient.

- A **real simple Lie algebra** $\mathfrak{k}$, namely the Lie algebra of K. This always has $\mathfrak{k} \otimes \mathbb{C} = \mathfrak{g}$. This weird-looking symbol '$\mathfrak{k}$' is a lower-case Gothic 'k'. You'll need to learn your Gothic letters to pass this course.

- A kind of 'incidence geometry' with G as its symmetry group, with one kind of 'geometrical figure' for each dot in the Dynkin diagram D. More on this later; this will answer the all-important question *what is really going on in this game?*

- A **finite reflection group** W — that is, a finite group of transformations of $\mathbb{R}^n$ generated by reflections, where n is the number of dots in D. We'll soon see how this gets into the game.

- A **lattice** $L \subseteq \mathbb{R}^n$ with W as its symmetry group — that is, a subset of the form

  $$L = \{k_1 v_1 + \cdots + k_n v_n \colon\ k_1, \ldots, k_n \in \mathbb{Z}\}$$

  where $v_1, \ldots, v_n$ are a basis of $\mathbb{R}^n$. If you think of $\mathbb{R}^n$ as a group with addition as the group operation, a lattice gives a subgroup isomorphic to $\mathbb{Z}^n$. However, not every subgroup isomorphic to $\mathbb{Z}^n$ is a lattice!

A Dynkin diagram also gives us a lot more, but you should already be impressed. So, let's start seeing how this stuff actually works.

2 A_2

Rather than going into the general theory, let's consider the simplest example, the A_n series of Dynkin diagrams — and then let's zoom in and look at the case A_2.

The Dynkin diagram A_n has n dots in a row, connected by edges. So, for example, here's A_3:

Here's A_2:

$$\bullet\!-\!\!-\!\!-\!\bullet$$

and here's A_1:

$$\bullet$$

Everyone seems to be scared of A_0, so let's not think about that.

The complex simply-connected simple Lie group corresponding to A_n is

$$G = \mathrm{SL}(n+1, \mathbb{C}) = \{(n+1) \times (n+1) \text{ complex matrices with determinant } = 1\}$$

Notice the obnoxious '+1'. There's no way around this; we could change our notation here but we'd just suffer somewhere else. You may wish to check that G is a complex Lie group, and simply connected, and simple. How easy this is depends on what you know.

Any complex Lie group has a maximal compact subgroup, which is a plain old real Lie group. Indeed, it usually has lots of them, but they're all conjugate to each other! For our particular G, everyone's favorite maximal compact subgroup is

$$K = \mathrm{SU}(n+1) = \{(n+1) \times (n+1) \text{ unitary complex matrices with determinant} = 1\}.$$

You may want to check that this is a Lie group. It's a real Lie group whose matrices have complex entries: remember that we say a Lie group is *complex* if its associated Lie algebra is a vector space over $\mathbb{C}$. $\mathrm{SU}(n+1)$ is compact because it's a closed bounded subset of the vector space of $(n+1) \times (n+1)$ complex matrices. It's bounded because each column of a matrix in $\mathrm{SU}(n+1)$ has norm less than or equal to 1.

But now let's get more specific and talk about the case $n = 2$. The case $n = 1$ is also incredibly important, but it's a bit too degenerate to illustrate the general pattern. Everything I say about A_2 will have straightforward generalizations to all the other A_n — and a bunch of what I say will generalize in a more clever way to *all* Dynkin diagrams. But it's easiest to see what's going on by starting with an example. At least, that's the philosophy of this course.

So, here we go. Take this Dynkin diagram:

$$\mathrm{A}_2 = \bullet\!-\!\!-\!\!-\!\bullet$$

What does this have to do with the complex simple Lie group

$$G = \mathrm{SL}(3, \mathbb{C}) = \{3 \times 3 \text{ complex matrices with determinant} = 1\}$$

or its maximal compact subgroup

$$K = \mathrm{SU}(3) = \{3 \times 3 \text{ unitary complex matrices with determinant} = 1\}?$$

The key is to look at the diagonal matrices. Diagonal matrices are great because they all commute and they're easy to multiply. The diagonal matrices in G form a subgroup

$$A = \{\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} : \ a, b, c \in \mathbb{C}, abc = 1\}$$

It's called A because it's *maximal abelian* subgroup of G — any matrix that commutes with everything in A has to itself be diagonal! And, since we've got 3 numbers satisfying one equation, the group A is a *2-dimensional* complex Lie group. (That's 4 real dimensions, but 2 complex dimensions.) As we'll see, this is why there are 2 dots in our Dynkin diagram! The number of dots is the complex dimension of the maximal abelian subgroup.

It's also great to look at the diagonal matrices in K. These form a subgroup

$$T = \{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} : \; a, b, c \in \mathrm{U}(1), abc = 1\}.$$

Note that now a, b and c have to be complex numbers of norm 1, to make the matrix unitary. So, they live in the unit circle, also known as

$$\mathrm{U}(1) = \{1 \times 1 \text{ unitary complex matrices}\}.$$

This group T is called T because it's a *maximal torus* in K. It's isomorphic to a torus, that is a product of copies of U(1):

$$T = \{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (ab)^{-1} \end{pmatrix} : \; a, b \in \mathrm{U}(1)\}.$$

Any compact Lie group (like K) has a maximal abelian subgroup — in fact a bunch of them, but they're all conjugate. And, this maximal abelian subgroup will be a torus. So, we call it a maximal torus.

Note that T is a 2-dimensional *real* Lie group. That's also no coincidence: the real dimension of T is the complex dimension of A. In fact there's a precise sense in which A is the 'complexification' of T. Similarly, G is the complexification of K.

But now for the really fun part. The Lie algebra of any abelian Lie group is itself abelian, meaning that the Lie bracket $[x, y]$ is zero for any pair of elements in the Lie algebra. In particular, the Lie algebra of our T is just

$$\mathfrak{t} = \{ \begin{pmatrix} ia & 0 & 0 \\ 0 & ib & 0 \\ 0 & 0 & ic \end{pmatrix} : \; a, b, c \in \mathbb{R}, a + b + c = 0\}$$

since it's precisely a guy x of this form that you can multiply by $t \in \mathbb{R}$ and exponentiate to get a 1-parameter family of guys $\exp(tx)$ in T. Exponentiating matrices takes work — you use the Taylor series for exp — but for diagonal matrices it's really easy: you just exponentiate each diagonal entry! So, if we take a guy in $\mathfrak{t}$, like

$$x = \begin{pmatrix} ia & 0 & 0 \\ 0 & ib & 0 \\ 0 & 0 & ic \end{pmatrix}$$

with $a, b, c \in \mathbb{R}, a + b + c = 0$, its exponential will be

$$\exp(x) = \begin{pmatrix} e^{ia} & 0 & 0 \\ 0 & e^{ib} & 0 \\ 0 & 0 & e^{ic} \end{pmatrix}$$

which lies in T! So, we have an exponential map

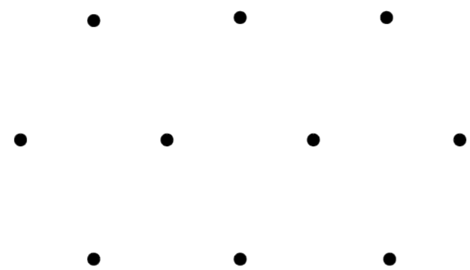
$$\exp \colon \mathfrak{t} \to T$$

This satisfies

$$\exp(a + b) = \exp(a) \exp(b)$$

Indeed, this equation is always true for *abelian* Lie algebras — for the nonabelian case this simple law becomes something more complicated, the 'Baker–Campbell–Hausdorff formula'.

Now, what's so fun about this exponential map? The point is that it lets us find a *lattice* in the vector space $\mathfrak{t}$, that is, a bunch of points in a grid something like this:



We started by studying some Lie groups, which are 'continuous' structures — smooth manifolds in fact. But this lattice is a 'discrete' structure! Understanding this lattice turns out to be crucial for understanding the Lie groups. It's this interplay of continuous and discrete that makes this subject fun.

But how do we actually get this lattice?

Thanks to the above equation, exp is a *group homomorphism* from $\mathfrak{t}$, thought of as a Lie group with addition as the group operation to T, thought of as a Lie group with multiplication as the group operations! (Don't get confused: $\mathfrak{t}$ is a Lie algebra, but any Lie algebra has an underlying vector space, and any vector space gives a Lie group if you use addition as the group operation.)

So, we can take the kernel of $\exp\colon \mathfrak{t} \to T$, which will be a subgroup of $\mathfrak{t}$. The kernel ker(exp) consists of all guys

$$x = \begin{pmatrix} ia & 0 & 0 \\ 0 & ib & 0 \\ 0 & 0 & ic \end{pmatrix}$$

such that $a, b, c \in \mathbb{R}$, $a + b + c = 0$, and most importantly:

$$\begin{pmatrix} e^{ia} & 0 & 0 \\ 0 & e^{ib} & 0 \\ 0 & 0 & e^{ic} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So,

$$\ker(\exp) = \{ \begin{pmatrix} ia & 0 & 0 \\ 0 & ib & 0 \\ 0 & 0 & ic \end{pmatrix} : \; a, b, c \in 2\pi\mathbb{Z}, a + b + c = 0\}$$

Let's check that this is really a lattice in $\mathfrak{t}$. Remember that a **lattice** is a subgroup of a vector space consisting of all integer linear combinations of some basis vectors. The vector space $\mathfrak{t}$ has a basis

$$2\pi B_1 = \begin{pmatrix} 2\pi i & 0 & 0 \\ 0 & -2\pi i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$2\pi B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\pi i & 0 \\ 0 & 0 & -2\pi i \end{pmatrix}$$

and ker(exp) consists of all linear combinations of these basis vectors:

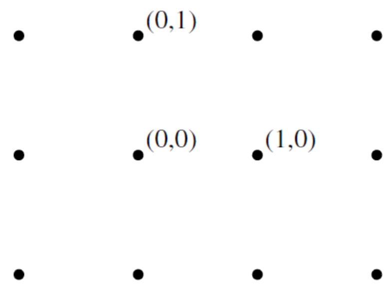
$$\ker(\exp) = \{2\pi(k_1 B_1 + k_2 B_2) \colon \; k_1, k_2 \in \mathbb{Z}\}.$$

So, it's indeed a lattice.

How can we *draw* this lattice? We could use our basis to set up Cartesian coordinates, and draw the point

$$2\pi(k_1 B_1 + k_2 B_2) \in \ker(\exp)$$

as the point (k_1, k_2) using these coordinates. Then we get a square-looking lattice, like this:



But *every* lattice in the plane looks square if we draw it using this method! It's not a very good approach. It's better to draw all the triples (a, b, c) with $a, b, c \in 2\pi\mathbb{Z}, a + b + c = 0$ as points in 3d space. They lie in the plane $a + b + c = 0$. And, if we look straight at this plane, we get a bunch of dots in a hexagonal array! The six points nearest to the origin are the corners of a regular hexagon:

$$\begin{array}{c} (2\pi, -2\pi, 0) \\ (-2\pi, 2\pi, 0) \\ (2\pi, 0, -2\pi) \\ (-2\pi, 0, 2\pi) \\ (0, 2\pi, -2\pi) \\ (0, -2\pi, 2\pi) \end{array}$$

At this point you're probably getting sick of all these 2π's — everyone does. So, people often define a new improved exponential function

$$e(x) = \exp(2\pi x)$$

and change their minds and define the lattice L by

$$L = \ker e \subseteq \mathfrak{t}.$$

In our A_2 example,

$$L = \{ \left(\begin{array}{ccc} ia & 0 & 0 \\ 0 & ib & 0 \\ 0 & 0 & ic \end{array} \right) : \; a, b, c \in \mathbb{Z}, a + b + c = 0\}$$

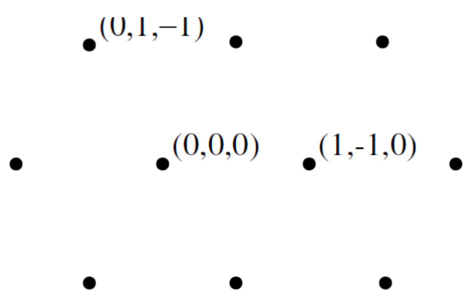
Now the six points closest to the origin are

$$\begin{array}{c} (i, -i, 0) \\ (-i, i, 0) \\ (i, 0, -i) \\ (-i, 0, i) \\ (0, i, -i) \\ (0, -i, i) \end{array}$$

If we're trying to *draw* these, we can ignore the factor of i and just draw these points in $\mathbb{R}^3$:

$$\begin{array}{c} (1,-1,0) \\ (-1,1,0) \\ (1,0,-1) \\ (-1,0,1) \\ (0,1,-1) \\ (0,-1,1) \end{array}$$

Regardless of these details, we call this lattice the A_2 **lattice**. It looks like this:



and it's an incredibly fundamental structure. If you're trying to pack pennies on the plane in the densest possible way, this is how you should place their centers! That may seem obvious after you try it for a while. But it's not trivial to prove. It was first proved here:

- László Fejes Tóth, Über einen geometrischen Satz, *Math. Z.* **46** (1940), 79–83.

If you pack pennies this way, about 91% of the plane will be covered. A square lattice only covers about 79%. Work it out yourself.

For us, what's most important is that *every* compact simple Lie group gives a lattice. We don't get every lattice this way: only certain highly symmetrical ones. And, we can reconstruct the groups from their lattices! Eventually, classifying the nice lattices will let us classify the compact simple Lie groups. Even better, the lattices hold tons of information about the groups!

When we drew the A_2 lattice 'correctly', we saw that it has 6-fold symmetry. Soon we'll see more precisely what counts as drawing the lattice 'correctly', and *why* the A_2 lattice has 6-fold symmetry. But for now, let's just go ahead and look at the A_3 lattice. Can you guess what its symmetries will be?