Lie Theory Through Examples

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Lecture 4

1 Classifying Unitary Representations: A_1

Last time we saw how to classify unitary representations of a torus T using its **weight lattice** L^*: the dual of the lattice L that's the kernel of the exponential map $e\colon \mathfrak{t} \to T$. Now we should study some examples. But first, a quick review:

Any point $\ell \in L^*$ gives a 1-dimensional representation of T

$$\rho_\ell\colon T \to \mathrm{U}(1)$$

with

$$\rho_\ell(e(x)) = e^{2\pi i \ell(x)}$$

for $x \in \mathfrak{t}$. We call this the **weight-ℓ representation**. This representation is irreducible and unitary.

Every irreducible unitary representations of T is unitarily equivalent to a weight-ℓ representation for some ℓ. Even better, *every* unitary representation of T is a big direct sum, where we take the direct sum of $d(\ell)$ copies of the weight-ℓ representation, and then the direct sum over all ℓ. So, we can describe a unitary representation of T by a function

$$d\colon L^* \to \mathbb{N}.$$

This function d deserves a snappy name, so let's call it the *weighting* of the representation. We call $d(\ell)$ the *multiplicity* of the weight ℓ.

More generally, if

$$\rho\colon K \to \mathrm{U}(H)$$

is a unitary representation of a compact simply-connected simple Lie group, we can restrict ρ to the maximal torus $T \subseteq K$ and then compute d as above. We then have the following amazing fact, which we will not prove here:

Theorem 1 *Two unitary representation of K are unitarily equivalent if and only if they have the same weighting*

$$d\colon L^* \to \mathbb{N}.$$

Now let's do some examples. First let's do the case of A_1 — an example that produces such a dull lattice that we skipped it on our first tour. This Dynkin diagram corresponds to $K = \mathrm{SU}(2)$. As usual, we get a maximal torus consisting of diagonal matrices:

$$T = \{ \left(\begin{array}{cc} a & 0 \\ 0 & b \end{array}\right) \colon\ a, b \in \mathrm{U}(1), ab = 1\}.$$

but now this is a 1-dimensional torus, isomorphic to $\mathrm{U}(1)$ as follows:

$$e^{i\theta} \mapsto \left(\begin{array}{cc} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{array}\right).$$

So, the Lie algebra $\mathfrak{t}$ of the maximal torus is isomorphic to $\mathbb{R}$, and if we think of it this way, the exponential map is

$$\begin{array}{ccc} e\colon \mathfrak{t} & \to & T \\ x & \mapsto & \left(\begin{array}{cc} e^{2\pi i x} & 0 \\ 0 & e^{-2\pi i x} \end{array}\right) \end{array}$$

So, the lattice L is just $\mathbb{Z} \subseteq \mathbb{R}$.

So, the $\mathbf{A_1}$ **lattice** is just the integers! Similarly, the dual $\mathfrak{t}^*$ is also isomorphic to $\mathbb{R}$, and the dual lattice L^* is also isomorphic to $\mathbb{Z}$.

But now let's take some unitary representation of SU(2) and see how it gives a map

$$d\colon \mathbb{Z} \to \mathbb{N}.$$

For example, let's try the representation where SU(2) acts on $\mathbb{C}^2$ in the obvious way. If we write an element of $\mathbb{C}^2$ as a column vector

$$\begin{pmatrix} a \\ b \end{pmatrix}$$

then for any $x \in \mathfrak{t}$, we have

$$e(x) = \begin{pmatrix} e^{2\pi i x} & 0 \\ 0 & e^{-2\pi i x} \end{pmatrix}$$

acts on it to give

$$\begin{pmatrix} e^{2\pi i x}\, a \\ e^{-2\pi i x}\, b \end{pmatrix}$$

Note that $\mathbb{C}^2$ is a direct sum of two irreducible representations of T. These subrepresentations are spanned by

$$z_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$z_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

respectively. Since

$$e(x)z_1 = e^{2\pi x} z_1$$

and

$$e(x)z_2 = e^{-2\pi x} z_2,$$

the weights of these irreps are 1 and -1, respectively. So, our representation corresponds to a weighting

$$d\colon L^* \to \mathbb{N}$$

that's zero except at ± 1, where it equals one. We can draw it like this:

The tiny dots are weights ℓ with multiplicity zero: $d(\ell) = 0$. We draw them just so we can see the whole weight lattice. The bigger dots are the weights with multiplicity 1.

Since SU(2) has a unitary representation on $\mathbb{C}^2$, it also has one on $S^n\mathbb{C}^2$, the nth symmetrized tensor power of $\mathbb{C}^2$. Elements of this space are degree-n homogeneous polynomials in two variables, say z_1 and z_2. When $n = 1$ we're back to the example we just saw, where

$$\begin{aligned} e(x)z_1 &= e^{2\pi x} z_1 \\ e(x)z_2 &= e^{-2\pi x} z_2 \end{aligned}$$

for $x \in \mathfrak{t}$. For general n, $S^n\mathbb{C}^2$ has a basis of monomials $z_1^p z_2^q$ where $p + q = n$. It's easy to check that

$$e(x) z_1^p z_2^q = e^{2\pi i(p-q)} z_1^p z_2^q,$$

so each of these monomials spans a 1-dimensional irreducible representation of $T \subseteq \mathrm{SU}(2)$. The weights of these representations are the numbers $p - q$, or in other words:

$$n, n-2, \ldots, 2-n, -n.$$

To draw this, draw the integers and then draw a single circle around the points from $-n$ to n, skipping every other one.

Here's the picture for $n = 2$:

As you can see, this is a 3-dimensional representation. In fact this representation is very important: it's equivalent to the representation where $g \in \mathrm{SU}(2)$ acts on a matrix $T \in \mathfrak{sl}(2, \mathbb{C})$ by:

$$\rho(g)T = gTg^{-1}$$

In fact,

$$\mathfrak{sl}(2, \mathbb{C}) \cong \mathbb{C} \otimes \mathfrak{su}(2)$$

and this representation is just the 'complexification' of the adjoint representation of SU(2), where it acts on its own Lie algebra.

Here's the picture for $n = 3$:

As you can see, this is a 4-dimensional representation. In general, the space $S^n\mathbb{C}$ gives an $(n + 1)$-dimensional representaiton of SU(2). It's irreducible, and in fact these are *all* the irreps of SU(2)! Physicists call the rep of SU(2) on $S^n\mathbb{C}$ the **spin-j representation**, where $j = n/2$.

2 SU(2) versus SO(3)

We're calling the Lie group SU(2) 'simple', but that doesn't mean it has no interesting normal subgroups! Remember, we say a Lie group is **simple** if all its normal subgroups are discrete — or equivalently, if its Lie algebra is simple. It's easy to see that the center of SU(2) consists of the matrices ± 1: this is a discrete normal subgroup, and in fact the only one except for the trivial subgroup.

So, we can form the quotient $\mathrm{SU}(2)/\{\pm 1\}$, and this will again be a simple Lie group. In fact, it's isomorphic to SO(3)! Remember:

Definition 1 *The* **orthogonal group** $\mathrm{O}(n)$ *is the group of all linear transformations* $g \colon \mathbb{R}^n \to \mathbb{R}^n$ *that preserve the usual inner product on* $\mathbb{R}^n$*:*

$$\langle gv, gw \rangle = \langle v, w \rangle$$

for all $v, w \in \mathbb{R}^n$. *Equivalently,*

$$\mathrm{O}(n) = \{g \in \mathrm{GL}(n, \mathbb{R})\colon \; g^* g = 1\}.$$

The **special orthogonal group** *is*

$$\mathrm{SO}(n) = \{g \in \mathrm{O}(n, \mathbb{R})\colon \; \det(g) = 1\}.$$

Any product of an odd number of reflections gives an element of $\mathrm{O}(n)$ that's not in $\mathrm{SO}(n)$, and in fact these are all such elements. Products of even numbers of reflections give all elements of $\mathrm{SO}(n)$.

Anyway, it's pretty easy to get a homomorphism

$$\rho\colon \mathrm{SU}(2) \to \mathrm{SO}(3).$$

Namely, we let $g \in \mathrm{SU}(2)$ act on the Lie algebra $\mathfrak{su}(2)$ by the **adjoint representation**

$$\mathrm{Ad}(g)v = gvg^{-1}.$$

This preserves the obvious inner product on $\mathfrak{su}(2)$, the one we've already seen:

$$\langle v, w\rangle = -\mathrm{tr}(vw).$$

Here's why:

$$\langle gv, gw\rangle = -\mathrm{tr}(gvg^{-1}gwg^{-1}) = -\mathrm{tr}(vw) = \langle v, w\rangle.$$

So, if we identify $\mathfrak{su}(2)$ with this inner product with $\mathbb{R}^3$ with its standard inner product, we can think of Ad as a homomorphism $\rho\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$.

It's easy to see that $\pm 1 \in \mathrm{SU}(2)$ are in the kernel of ρ, since they commute with all 2×2 matrices. In fact it's easy to check that only scalar multiples of the identity operator can commute with everyone in $\mathfrak{su}(2)$ — just assume a matrix commutes with all three Pauli matrices, and see what it must be like. So, the kernel of ρ is *exactly* $\{\pm 1\}$. It's also easy to check:

Exercise 1 *The homomorphism $\rho\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ is onto.*

So,

$$\mathrm{SU}(2)/\{\pm 1\} \cong \mathrm{SO}(3).$$

We say SU(2) is a **double cover** of SO(3).

Now, if we take a maximal torus $T' \subseteq \mathrm{SU}(2)$, and map it to SO(3) via ρ, it gets sent to a maximal torus $T \subseteq \mathrm{SU}(2)$. Because ρ is 2-1, we can think of T and T' as having the same Lie algebra $\mathfrak{t}$, but with the map

$$\mathfrak{t}/L \cong T \xrightarrow{\rho} T' \cong \mathfrak{t}/L'$$

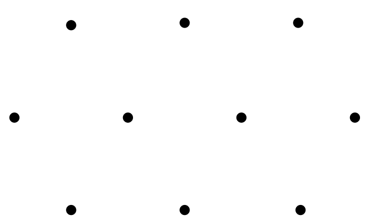
being 2-1. This means $L \subset L'$ — and indeed L must be a lattice with *half the density* of L'.

This, in turn, means that L'^* is a sublattice of L^* — with half the density of L^*! In the previous section, we showed how to think of the weight lattice L^* of SU(2) as the integers, $\mathbb{Z}$. So, in this picture, the weight lattice of SO(3) consists of the even integers, $2\mathbb{Z}$.

Any representation of SU(2) coming from a representation of SO(3) must have its weights lying in this sublattice. So, if we look at our results from the previous section, we can guess that the rep of SU(2) on $S^n(\mathbb{C}^2)$ comes from a representation of SO(3) when n is *even*. And it's true.

3 Classifying Unitary Representations: A_2

We know the lattice L for A_2:



What does L^* look like? We can cheat and use our inner product on A_2 to identify the vector space $\mathfrak{t}$ containing L with its dual vector space. Then the dual lattice L^* looks hexagonal, a lot like L... but beware, it's not the *same* hexagonal lattice.

Exercise 2 *Draw L^* and L in the same picture.*

Exercise 3 *Draw the weighting $d\colon L^* \to \mathbb{N}$ for the obvious representation of* $\mathrm{SU}(3)$ *on* $\mathbb{C}^3$ *— the so-called* **tautologous representation**.

Exercise 4 *Draw the weighting $d\colon L^* \to \mathbb{N}$ for the dual of the tautologous representation of* $\mathrm{SU}(3)$*, on* $(\mathbb{C}^3)^*$.

Exercise 5 *Draw the weighting $d\colon L^* \to \mathbb{N}$ for the tensor product of the above two representations of* $\mathrm{SU}(3)$*. Hint: use the following exercise.*

Exercise 6 *Suppose ρ and σ are unitary representations of a simply-connected compact simple Lie group K. Let*

$$d_\rho, d_\sigma \colon L^* \to \mathbb{N}$$

be the corresponding functions. Show that

$$d_{\rho\otimes\sigma} = d_\rho * d_\sigma$$

where the **convolution product** $*$ *is defined by*

$$(f * g)(\ell) = \sum_{\ell' + \ell'' = \ell} f(\ell') g(\ell'')$$

The tensor product $\mathbb{C}^3 \otimes (\mathbb{C}^3)^*$ is isomorphic to the space of 3×3 matrices, which becomes a representation of $\mathrm{SU}(3)$ via

$$\rho(g)T = gTg^{-1}$$

for $T\colon \mathbb{C}^3 \to \mathbb{C}^3$. This representation has a 1-dimensional subrepresentation consisting of multiples of the identity matrix. Indeed, it's the direct sum of this 1d rep and an 8-dimensional subrepresentation that consists of the traceless matrices:

$$\mathbb{C}^3 \otimes (\mathbb{C}^3)^* \cong \mathbb{C} \oplus \mathfrak{sl}(3, \mathbb{C}).$$

Exercise 7 *Draw the function $d\colon L^* \to \mathbb{N}$ for the the above representations of* $\mathrm{SU}(3)$ *on* $\mathbb{C}$ *and* $\mathfrak{sl}(3, \mathbb{C})$*. Hint: use the following exercise.*

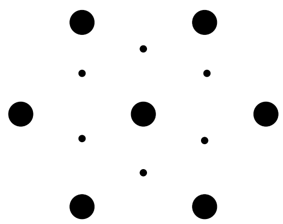
Exercise 8 *Suppose ρ and σ are unitary representations of a simply-connected compact simple Lie group K. Let*

$$d_\rho, d_\sigma \colon L^* \to \mathbb{N}$$

be the corresponding functions. Show that

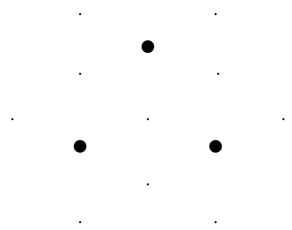
$$d_{\rho\oplus\sigma} = d_\rho + d_\sigma.$$

Your answer to Exercise 2 should look a bit like this:

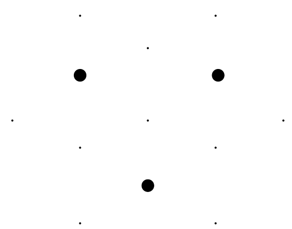


The big dots are in the lattice L, while the small ones *and* the big ones are in L^*. It's easy to see that $L \subseteq L^*$, since the inner product of any two vectors in L is an integer.

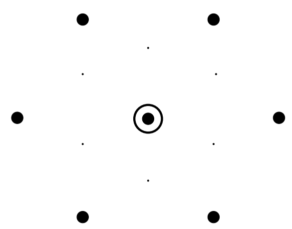
Your picture of the weighting for the tautologous representation of SU(3) should look a bit like this:



Here the tiny dots are weights with multiplicity 0, while the bigger ones have multiplicity 1. Similarly, your picture of the weighting for the *dual* of the tautologous representation should look a bit like this:



If your pictures look rotated or upside-down compared to mine, that's no big deal: it's just an arbitrary convention. Finally, your picture of the weighting for the representation of SU(3) on $\mathfrak{sl}(3, \mathbb{C})$ should look a bit like this:



Here the tiny dots have multiplicity 0, the bigger ones has multiplicity 1, and the one with a circle around it has multiplicity 2. If we get a weight with an even larger multiplicity, we can just draw more circles around it!

In general, the weighting for an irreducible representation of SU(3) will look like this. First, draw a big hexagon centered at the origin with edge lengths a, b, a, b, a, b. The multiplicity is 1 for weights on the the edge of the hexagon, 2 around the next hexagon inside, and so on, until the hexagon degenerates to a triangle. At that point the multiplicity is constant, namely $\min(a, b) + 1$. The triangle can be either 'right-side up' or 'upside-down' — as we've seen in the tautologous rep and its dual.