Lie Theory Through Examples

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Lecture 3

1 Representations of Lie Groups

We've been having some fun getting lattices from simply-connected complex simple Lie groups. If someone hands us one of these, say G, we first choose a maximal compact subgroup $K \subseteq G$. They're all conjugate inside G, so it doesn't matter which one we choose. Then, we choose a maximal torus $T \subseteq K$. Again, they're all conjugate inside K, so it doesn't matter which one we choose. Then we can work out the Lie algebra $\mathfrak{t}$ of T and find a lattice

$$L = \ker e = \subseteq \mathfrak{t},$$

where

$$\begin{array}{rcl} e\colon \mathfrak{t} & \to & T \\ x & \mapsto & \exp(2\pi x). \end{array}$$

So far we've done this for $G = \mathrm{SL}(3, \mathbb{C})$ and $K = \mathrm{SL}(4, \mathbb{C})$. The other $\mathrm{SL}(n, \mathbb{C})$'s work similarly, and soon we'll do even more examples. It's a lot of fun.

But, *what's it all good for?*

Among other things, it's good for classifying the complex-analytic representations of G, and the unitary representations of K. So, we need a word or two about these.

Remember that a **representation** of any Lie group H is a smooth homomorphism

$$\rho\colon H \to \mathrm{GL}(V)$$

where V is a vector space and $\mathrm{GL}(V)$ is the group of invertible linear transformations of V. In what follows we'll always assume V is finite-dimensional. When $V = \mathbb{C}^n$ we also call $\mathrm{GL}(V)$ the **general linear group** $\mathrm{GL}(n, \mathbb{C})$.

Now, the group $\mathrm{GL}(V)$ is always a complex manifold: we can cover it with coordinate charts that look like $\mathbb{C}^n$, with complex-analytic transition functions. It makes sense to talk about complex-analytic maps between complex manifolds. And indeed, $\mathrm{GL}(V)$ is a **complex Lie group**: a complex manifold where the functions describing multiplication and inverses are complex analytic. To see this, just use the usual formulas for multiplying and taking inverses of matrices.

If H is a complex Lie group, we say a representation $\rho\colon H \to \mathrm{GL}(V)$ is **complex-analytic** if it is complex-analytic as a map between complex manifolds. Such representations are easy to come by:

Exercise 1 *Show that* $\mathrm{SL}(n, \mathbb{C})$ *is a complex Lie group, and the obvious representation of* $\mathrm{SL}(n, \mathbb{C})$ *on* $\mathbb{C}^n$ *is complex analytic.*

Exercise 2 *Show that if* $\rho, \sigma\colon G \to \mathrm{GL}(V)$ *are complex-analytic representations, so are* $\rho \oplus \sigma$ *and* $\rho \otimes \sigma$*.*

Exercise 3 *Show that any subrepresentation of a complex-analytic representation is complex-analytic.*

On the other hand, unitary representations are also nice. Given a Lie group H and a finite-dimensional Hilbert space $\mathcal{H}$, we define a **unitary representation** to be a smooth homomorphism:

$$\rho\colon K \to \mathrm{U}(\mathcal{H})$$

where $\mathrm{U}(\mathcal{H})$ is the group of unitary operators on a Hilbert space H. When $H = \mathbb{C}^n$ we also call $\mathrm{U}(H)$ the **unitary group** $\mathrm{U}(n)$.

Compact Lie groups have lots of unitary representations:

Exercise 4 *Suppose* $\rho\colon K \to \mathrm{GL}(V)$ *is a (finite-dimensional) representation of a compact Lie group. Show that there is an inner product on* V *that is invariant under* ρ*, so that letting* $\mathcal{H}$ *denote* V *made into a Hilbert space with this inner product, we have*

$$\rho\colon K \to U(\mathcal{H}).$$

The exercises we saw for complex-analytic representations all have analogues for unitary representations:

Exercise 5 *Show that* $\mathrm{SU}(n)$ *is a compact Lie group, and the obvious representation of* $\mathrm{SU}(n)$ *on* $\mathbb{C}^n$ *is unitary.*

Exercise 6 *Show that if* $\rho, \sigma\colon G \to \mathrm{GL}(V)$ *are unitary representations, so are* $\rho \oplus \sigma$ *and* $\rho \otimes \sigma$*.*

Exercise 7 *Show that any subrepresentation of a unitary representation is unitary.*

But, the really cool part is that when G is a complex simple Lie group and K is its maximal compact subgroup, the *complex-analytic* representations of G correspond in a one-to-way to *unitary* representations of K. This fact was called the **unitarian trick** by Hermann Weyl, who used it to do great things. Let's state it a bit more precisely:

Theorem 1 *Suppose* G *is a complex simple Lie group and* K *is its maximal compact subgroup. Given a (finite-dimensional) complex-analytic representation*

$$\rho\colon G \to \mathrm{GL}(V),$$

there exists an inner product on V *making* $\rho|_K$ *into a unitary representation. Conversely, given a (finite-dimensional) unitary representation*

$$\rho\colon K \to \mathrm{U}(H),$$

there exists a unique extension of ρ *to a complex-analytic representation of* G *on the vector space* H*.*

2 The Weight Lattice

Now say we have our favorite kind of Lie group G: a simply-connected complex simple Lie group. Say someone hands us a complex-analytic representation of G. We want to understand it and classify it. By the above theorem, we can think of it as a unitary representation of the maximal compact K without losing any information.

So, let's do that: say we have unitary representation ρ of K on a finite-dimensional Hilbert space H. What do we do now? Since the maximal torus T is a subgroup of K, we get a unitary representation of T:

$$\rho|_T\colon T \to \mathrm{U}(H).$$

And in fact, the maximal torus is big enough that we can completely recover ρ from $\rho|_T$. This is not supposed to be obvious! But it's *great*, because unitary representations of tori are incredibly easy to understand.

Here's how we understand them. Any unitary representation of a torus

$$\alpha\colon T \to \mathrm{U}(H)$$

can be composed with the exponential map to give a unitary representation of the vector space $\mathfrak{t}$, thought of as a group:

$$\beta = \alpha e.$$

Clearly β is trivial on the kernel of e:

$$\beta(x) = 1 \text{ for all } x \in L.$$

Conversely, it's easy to see a representation

$$\beta\colon \mathfrak{t} \to \mathrm{U}(H)$$

comes from a representation α as above iff β is trivial on L. Even better, since the exponential map is onto, knowing β tells us α. It's also easy to see that β is irreducible iff α is.

So, we just need to understand unitary representations

$$\beta\colon \mathfrak{t} \to \mathrm{U}(H)$$

that are trivial on L. Let's start with irreducible ones. By Schur's Lemma, an irreducible representation of an abelian group is 1-dimensional. In this case, $\beta(x)$ is just multiplication by some unit complex number; to be a representation we also need

$$\beta(x+y) = \beta(x)\beta(y).$$

So, it's easy to see that we get all *irreducible* unitary representations of $\mathfrak{t}$ from elements ℓ of the dual vector space $\mathfrak{t}^*$, like this:

$$\beta(x) = e^{2\pi i \ell(x)}.$$

For this to be trivial on L we need

$$\ell(x) \in \mathbb{Z} \text{ for all } x \in L.$$

This means that ℓ needs to lie in the set

$$L^* = \{\ell \in \mathfrak{t}^* :\ \ell(x) \in \mathbb{Z} \text{ for all } x \in L\}.$$

In fact L^* is a lattice in $\mathfrak{t}^*$ just as L is a lattice in $\mathfrak{t}$! We call L^* the **dual lattice** of L.

Summarizing what we've seen so far:

Theorem 2 *An irreducible unitary representation α of a torus T is specified by choosing a point ℓ in the dual lattice L^*.*

We call the point $\ell \in L^*$ the **weight** of the representation. When T is the maximal torus of a simply-connected compact simple Lie group K, we call L^* the **weight lattice** of K.

But what if α fails to be irreducible? Then it's a *direct sum* of irreducible representations — and in an essentially unique way. To specify each of these irreducible representations, we pick a point in L^*. We can pick the same point more than once. But we need to pick just finitely many points, since H is finite-dimensional. So:

Theorem 3 *Finite-dimensional unitary representation ρ of a torus T are classified up to unitary equivalence by maps*

$$d\colon L^* \to \mathbb{N}$$

such that

$$\sum_{\ell \in L^*} d(\ell) < \infty.$$

The map d says how many times each irreducible unitary representation of T shows up in ρ. The theorem says that given any map $d\colon L^* \to \mathbb{N}$ with $\sum_{\ell \in L^*} d(\ell) < \infty$, there exists a unitary representation ρ with this d. Moreover, it says that two unitary representations of T are unitarily equivalent iff they have the same d. Here we say $\rho\colon K \to \mathrm{U}(H)$ and $\rho'\colon K \to \mathrm{U}(H')$ are **unitarily equivalent** if there is unitary operator $U\colon H \to H'$ such that

$$\rho'(k)U = U\rho(k)$$

for all $k \in K$.

For example, say we have a unitary representation of the circle group:

$$\alpha\colon \mathrm{U}(1) \to \mathrm{U}(H).$$

Here $\mathfrak{u}(1) \cong \mathbb{R}$ and $L \subseteq \mathbb{R}$ is just the set of integers. So, to specify a unitary irreducible representation of $\mathrm{U}(1)$ we pick an integer ℓ. Concretely, it goes like this:

$$\alpha(e^{i\theta}) = e^{i\ell\theta}.$$

In general, unitary representations of $\mathrm{U}(1)$ are classified by maps

$$d\colon \mathbb{Z} \to \mathbb{N}$$

as in the theorem.