

Lie Theory Through Examples

John Baez

Lecture 4

1 Classifying Unitary Representations: A_1

Last time we saw how to classify unitary representations of a torus T using its **weight lattice** L^* : the dual of the lattice L that's the kernel of the exponential map $e: \mathfrak{t} \rightarrow T$. Now we should study some examples. But first, a quick review:

Any point $\ell \in L^*$ gives a 1-dimensional representation of T

$$\rho_\ell: T \rightarrow \mathrm{U}(1)$$

with

$$\rho_\ell(e(x)) = e^{2\pi i \ell(x)}$$

for $x \in \mathfrak{t}$. We call this the **weight- ℓ representation**. This representation is irreducible and unitary.

Every irreducible unitary representation of T is unitarily equivalent to a weight- ℓ representation for some ℓ . Even better, *every* unitary representation of T is a big direct sum, where we take the direct sum of $d(\ell)$ copies of the weight- ℓ representation, and then the direct sum over all ℓ . So, we can describe a unitary representation of T by a function

$$d: L^* \rightarrow \mathbb{N}.$$

This function d deserves a snappy name, so let's call it the *weighting* of the representation. We call $d(\ell)$ the *multiplicity* of the weight ℓ .

More generally, if

$$\rho: K \rightarrow \mathrm{U}(\mathcal{H})$$

is a unitary representation of a compact simply-connected simple Lie group, we can restrict ρ to the maximal torus $T \subseteq K$ and then compute d as above. We then have the following amazing fact, which we will not prove here:

Theorem 1 *Two unitary representations of K are unitarily equivalent if and only if they have the same weighting*

$$d: L^* \rightarrow \mathbb{N}.$$

Now let's do some examples. First let's do the case of A_1 — an example that produces such a dull lattice that we skipped it on our first tour. This Dynkin diagram corresponds to $K = \mathrm{SU}(2)$. As usual, we get a maximal torus consisting of diagonal matrices:

$$T = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathrm{U}(1), ab = 1 \right\}.$$

but now this is a 1-dimensional torus, isomorphic to $\mathrm{U}(1)$ as follows:

$$e^{i\theta} \mapsto \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

So, the Lie algebra \mathfrak{t} of the maximal torus is isomorphic to \mathbb{R} , and if we think of it this way, the exponential map is

$$\begin{aligned} e: \mathfrak{t} &\rightarrow T \\ x &\mapsto \begin{pmatrix} e^{2\pi i x} & 0 \\ 0 & e^{-2\pi i x} \end{pmatrix} \end{aligned}$$

So, the lattice L is just $\mathbb{Z} \subseteq \mathbb{R}$.

So, the **A₁ lattice** is just the integers! Similarly, the dual \mathfrak{t}^* is also isomorphic to \mathbb{R} , and the dual lattice L^* is also isomorphic to \mathbb{Z} .

But now let's take some unitary representation of $SU(2)$ and see how it gives a map

$$d: \mathbb{Z} \rightarrow \mathbb{N}.$$

For example, let's try the representation where $SU(2)$ acts on \mathbb{C}^2 in the obvious way. If we write an element of \mathbb{C}^2 as a column vector

$$\begin{pmatrix} a \\ b \end{pmatrix}$$

then for any $x \in \mathfrak{t}$, we have

$$e(x) = \begin{pmatrix} e^{2\pi ix} & 0 \\ 0 & e^{-2\pi ix} \end{pmatrix}$$

acts on it to give

$$\begin{pmatrix} e^{2\pi ix} a \\ e^{-2\pi ix} b \end{pmatrix}$$

Note that \mathbb{C}^2 is a direct sum of two irreducible representations of T . These subrepresentations are spanned by

$$z_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$z_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

respectively. Since

$$e(x)z_1 = e^{2\pi x} z_1$$

and

$$e(x)z_2 = e^{-2\pi x} z_2,$$

the weights of these irreps are 1 and -1 , respectively. So, our representation corresponds to a weighting

$$d: L^* \rightarrow \mathbb{N}$$

that's zero except at ± 1 , where it equals one. We can draw it like this:



The tiny dots are weights ℓ with multiplicity zero: $d(\ell) = 0$. We draw them just so we can see the whole weight lattice. The bigger dots are the weights with multiplicity 1.

Since $SU(2)$ has a unitary representation on \mathbb{C}^2 , it also has one on $S^n \mathbb{C}^2$, the n th symmetrized tensor power of \mathbb{C}^2 . Elements of this space are degree- n homogeneous polynomials in two variables, say z_1 and z_2 . When $n = 1$ we're back to the example we just saw, where

$$\begin{aligned} e(x)z_1 &= e^{2\pi x} z_1 \\ e(x)z_2 &= e^{-2\pi x} z_2 \end{aligned}$$

for $x \in \mathfrak{t}$. For general n , $S^n \mathbb{C}^2$ has a basis of monomials $z_1^p z_2^q$ where $p + q = n$. It's easy to check that

$$e(x)z_1^p z_2^q = e^{2\pi i(p-q)x} z_1^p z_2^q,$$

so each of these monomials spans a 1-dimensional irreducible representation of $T \subseteq \text{SU}(2)$. The weights of these representations are the numbers $p - q$, or in other words:

$$n, n - 2, \dots, 2 - n, -n.$$

To draw this, draw the integers and then draw a single circle around the points from $-n$ to n , skipping every other one.

Here's the picture for $n = 2$:



As you can see, this is a 3-dimensional representation. In fact this representation is very important: it's equivalent to the representation where $g \in \text{SU}(2)$ acts on a matrix $T \in \mathfrak{sl}(2, \mathbb{C})$ by:

$$\rho(g)T = gTg^{-1}$$

In fact,

$$\mathfrak{sl}(2, \mathbb{C}) \cong \mathbb{C} \otimes \mathfrak{su}(2)$$

and this representation is just the 'complexification' of the adjoint representation of $\text{SU}(2)$, where it acts on its own Lie algebra.

Here's the picture for $n = 3$:



As you can see, this is a 4-dimensional representation. In general, the space $S^n\mathbb{C}$ gives an $(n + 1)$ -dimensional representation of $\text{SU}(2)$. It's irreducible, and in fact these are *all* the irreps of $\text{SU}(2)$! Physicists call the rep of $\text{SU}(2)$ on $S^n\mathbb{C}$ the **spin- j representation**, where $j = n/2$.

2 SU(2) versus SO(3)

We're calling the Lie group $\text{SU}(2)$ 'simple', but that doesn't mean it has no interesting normal subgroups! Remember, we say a Lie group is **simple** if all its normal subgroups are discrete — or equivalently, if its Lie algebra is simple. It's easy to see that the center of $\text{SU}(2)$ consists of the matrices ± 1 : this is a discrete normal subgroup, and in fact the only one except for the trivial subgroup.

So, we can form the quotient $\text{SU}(2)/\{\pm 1\}$, and this will again be a simple Lie group. In fact, it's isomorphic to $\text{SO}(3)$! Remember:

Definition 1 The **orthogonal group** $\text{O}(n)$ is the group of all linear transformations $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$ that preserve the usual inner product on \mathbb{R}^n :

$$\langle gv, gw \rangle = \langle v, w \rangle$$

for all $v, w \in \mathbb{R}^n$. Equivalently,

$$\text{O}(n) = \{g \in \text{GL}(n, \mathbb{R}): g^*g = 1\}.$$

The **special orthogonal group** is

$$\text{SO}(n) = \{g \in \text{O}(n, \mathbb{R}): \det(g) = 1\}.$$

Any product of an odd number of reflections gives an element of $O(n)$ that's not in $SO(n)$, and in fact these are all such elements. Products of even numbers of reflections give all elements of $SO(n)$.

Anyway, it's pretty easy to get a homomorphism

$$\rho: \mathrm{SU}(2) \rightarrow \mathrm{SO}(3).$$

Namely, we let $g \in \mathrm{SU}(2)$ act on the Lie algebra $\mathfrak{su}(2)$ by the **adjoint representation**

$$\mathrm{Ad}(g)v = gvg^{-1}.$$

This preserves the obvious inner product on $\mathfrak{su}(2)$, the one we've already seen:

$$\langle v, w \rangle = -\mathrm{tr}(vw).$$

Here's why:

$$\langle gv, gw \rangle = -\mathrm{tr}(gvg^{-1}gwg^{-1}) = -\mathrm{tr}(vw) = \langle v, w \rangle.$$

So, if we identify $\mathfrak{su}(2)$ with this inner product with \mathbb{R}^3 with its standard inner product, we can think of Ad as a homomorphism $\rho: \mathrm{SU}(2) \rightarrow \mathrm{SO}(3)$.

It's easy to see that $\pm 1 \in \mathrm{SU}(2)$ are in the kernel of ρ , since they commute with all 2×2 matrices. In fact it's easy to check that only scalar multiples of the identity operator can commute with everyone in $\mathfrak{su}(2)$ — just assume a matrix commutes with all three Pauli matrices, and see what it must be like. So, the kernel of ρ is *exactly* $\{\pm 1\}$. It's also easy to check:

Exercise 1 *The homomorphism $\rho: \mathrm{SU}(2) \rightarrow \mathrm{SO}(3)$ is onto.*

So,

$$\mathrm{SU}(2)/\{\pm 1\} \cong \mathrm{SO}(3).$$

We say $\mathrm{SU}(2)$ is a **double cover** of $\mathrm{SO}(3)$.

Now, if we take a maximal torus $T' \subseteq \mathrm{SU}(2)$, and map it to $\mathrm{SO}(3)$ via ρ , it gets sent to a maximal torus $T \subseteq \mathrm{SO}(3)$. Because ρ is 2-1, we can think of T and T' as having the same Lie algebra \mathfrak{t} , but with the map

$$\mathfrak{t}/L \cong T \xrightarrow{\rho} T' \cong \mathfrak{t}/L'$$

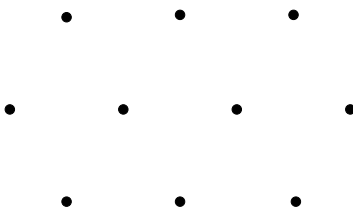
being 2-1. This means $L \subset L'$ — and indeed L must be a lattice with *half the density* of L' .

This, in turn, means that L'^* is a sublattice of L^* — with half the density of L^* ! In the previous section, we showed how to think of the weight lattice L^* of $\mathrm{SU}(2)$ as the integers, \mathbb{Z} . So, in this picture, the weight lattice of $\mathrm{SO}(3)$ consists of the even integers, $2\mathbb{Z}$.

Any representation of $\mathrm{SU}(2)$ coming from a representation of $\mathrm{SO}(3)$ must have its weights lying in this sublattice. So, if we look at our results from the previous section, we can guess that the rep of $\mathrm{SU}(2)$ on $S^n(\mathbb{C}^2)$ comes from a representation of $\mathrm{SO}(3)$ when n is *even*. And it's true.

3 Classifying Unitary Representations: A_2

We know the lattice L for A_2 :



What does L^* look like? We can cheat and use our inner product on A_2 to identify the vector space \mathfrak{t} containing L with its dual vector space. Then the dual lattice L^* looks hexagonal, a lot like L ... but beware, it's not the *same* hexagonal lattice.

Exercise 2 Draw L^* and L in the same picture.

Exercise 3 Draw the weighting $d: L^* \rightarrow \mathbb{N}$ for the obvious representation of $SU(3)$ on \mathbb{C}^3 — the so-called **tautologous representation**.

Exercise 4 Draw the weighting $d: L^* \rightarrow \mathbb{N}$ for the dual of the tautologous representation of $SU(3)$, on $(\mathbb{C}^3)^*$.

Exercise 5 Draw the weighting $d: L^* \rightarrow \mathbb{N}$ for the tensor product of the above two representations of $SU(3)$. *Hint: use the following exercise.*

Exercise 6 Suppose ρ and σ are unitary representations of a simply-connected compact simple Lie group K . Let

$$d_\rho, d_\sigma: L^* \rightarrow \mathbb{N}$$

be the corresponding functions. Show that

$$d_{\rho \otimes \sigma} = d_\rho * d_\sigma$$

where the **convolution product** $*$ is defined by

$$(f * g)(\ell) = \sum_{\ell' + \ell'' = \ell} f(\ell')g(\ell'')$$

The tensor product $\mathbb{C}^3 \otimes (\mathbb{C}^3)^*$ is isomorphic to the space of 3×3 matrices, which becomes a representation of $SU(3)$ via

$$\rho(g)T = gTg^{-1}$$

for $T: \mathbb{C}^3 \rightarrow \mathbb{C}^3$. This representation has a 1-dimensional subrepresentation consisting of multiples of the identity matrix. Indeed, it's the direct sum of this 1d rep and an 8-dimensional subrepresentation that consists of the traceless matrices:

$$\mathbb{C}^3 \otimes (\mathbb{C}^3)^* \cong \mathbb{C} \oplus \mathfrak{sl}(3, \mathbb{C}).$$

Exercise 7 Draw the function $d: L^* \rightarrow \mathbb{N}$ for the the above representations of $SU(3)$ on \mathbb{C} and $\mathfrak{sl}(3, \mathbb{C})$. *Hint: use the following exercise.*

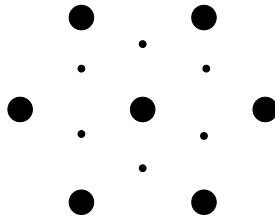
Exercise 8 Suppose ρ and σ are unitary representations of a simply-connected compact simple Lie group K . Let

$$d_\rho, d_\sigma: L^* \rightarrow \mathbb{N}$$

be the corresponding functions. Show that

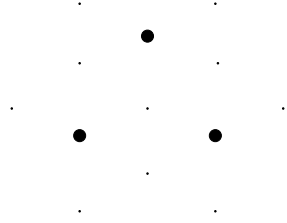
$$d_{\rho \oplus \sigma} = d_\rho + d_\sigma.$$

Your answer to Exercise 2 should look a bit like this:

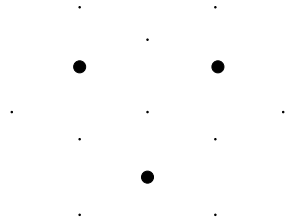


The big dots are in the lattice L , while the small ones *and* the big ones are in L^* . It's easy to see that $L \subseteq L^*$, since the inner product of any two vectors in L is an integer.

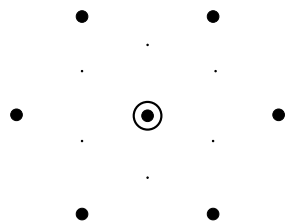
Your picture of the weighting for the tautologous representation of $SU(3)$ should look a bit like this:



Here the tiny dots are weights with multiplicity 0, while the bigger ones have multiplicity 1. Similarly, your picture of the weighting for the *dual* of the tautologous representation should look a bit like this:



If your pictures look rotated or upside-down compared to mine, that's no big deal: it's just an arbitrary convention. Finally, your picture of the weighting for the representation of $SU(3)$ on $\mathfrak{sl}(3, \mathbb{C})$ should look a bit like this:



Here the tiny dots have multiplicity 0, the bigger ones has multiplicity 1, and the one with a circle around it has multiplicity 2. If we get a weight with an even larger multiplicity, we can just draw more circles around it!

In general, the weighting for an irreducible representation of $SU(3)$ will look like this. First, draw a big hexagon centered at the origin with edge lengths a, b, a, b, a, b . The multiplicity is 1 for weights on the the edge of the hexagon, 2 around the next hexagon inside, and so on, until the hexagon degenerates to a triangle. At that point the multiplicity is constant, namely $\min(a, b) + 1$. The triangle can be either 'right-side up' or 'upside-down' — as we've seen in the tautologous rep and its dual.