Duality

Every category C has an opposite C^{op}. C^{op} has the same objects as C, but the morphisms are "turned around". So there's a 1-1 correspondence between morphisms in C & morphisms in C^{op}, with $f: x \to y$ in C corresponding to a morphism $f^{op}: y \to x$ in C^{op}. We compose morphisms in C^{op} by: $f^{op} \circ g^{op} = (g \circ f)^{op}$.

The study of how categories C relate to their partners C^{op} is called duality.

Note: $(C^{op})^{op} = C$. Just like for finite-dimensional vector spaces, $(V^*)^* \cong V$ via a natural isomorphism.

It turns out that THE DUAL OF GEOMETRY IS ALGEBRA.

In geometry we study "points"; in algebra we study addition & multiplication.

Descartes realized we can reduce (a lot of) geometry to algebra through "analytic geometry".

We can associate to any finite dimensional vector space over $\mathbb{R}$ a commutative ring $\mathcal{O}(V)$ consisting of all polynomial functions on V. If $V = \mathbb{R}^n$, the algebra $\mathcal{O}(V)$ consists of polynomials in the coordinate functions $X_1, \ldots, X_n$: $\mathcal{O}(V) = \mathbb{R}[X_1, \ldots, X_n]$.

So we go from a "space" V (a bunch of points) to an algebra $\mathcal{O}(V)$.

Then we can describe certain subspaces X of V: $X \xrightarrow{1\text{-}1} V$ as quotient algebras $\mathcal{O}(V)$: $\mathcal{O}(V) \xrightarrow{\text{onto}} \mathcal{O}(X) = \mathcal{O}(V)/I$ for an ideal I.

Example: the unit circle is a subspace of the plane: $S^1 \hookrightarrow \mathbb{R}^2$, where $S^1 = \{(x,y) : x^2+y^2-1 = 0\}$. Then there's an algebra $\mathcal{O}(S^1)$ of polynomials on S^1 with $\mathcal{O}(S^1) = \mathbb{R}[x,y]/\langle x^2+y^2-1\rangle$. So the 1-1 map $S^1 \to \mathbb{R}^2$ gets turned around, giving $\mathcal{O}(\mathbb{R}^2) \to \mathcal{O}(S^1)$ which is just restriction $f \mapsto f|_{S^1}$. Moreover $f, g \in \mathcal{O}(\mathbb{R}^2)$ restrict to the same function on S^1 iff $f - g \in \langle x^2+y^2-1\rangle$.

Algebraic geometry is the study of geometry using commutative rings.

Our idea is: subspaces of V should correspond to quotient rings of $\mathcal{O}(V)$, or ideals of $\mathcal{O}(V)$.

Problems:

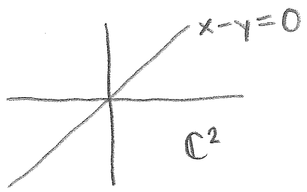
1) What about $\langle x^2+y^2+1\rangle \subseteq \mathcal{O}(\mathbb{R}^2)$?

The function x^2+y^2+1 doesn't vanish on $\mathbb{R}^2$, so it seems the corresponding subspace of $\mathbb{R}^2$ is $\emptyset$.

But there's another, simpler ideal corresponding to $\emptyset \in \mathbb{R}^2$, namely $\langle 1\rangle$. So $\emptyset = \{(x,y) : 1 = 0\}$, but we're getting 2 ideals corresponding to the same subspace. One way out is to use $\mathbb{C}$ instead of $\mathbb{R}$.

2) Alas, using $\mathbb{C}$ doesn't completely fix the problem of 2 different ideals corresponding to the same subspace.

There's a complex line in $\mathbb{C}^2$ given by $x=y$, with ideal $\langle x-y\rangle \subseteq \mathbb{C}[x,y]$.



But $(x-y)^2$ also vanishes only on this line, so we're getting a different ideal defining the same subspace: $\langle (x-y)^2\rangle \subseteq \mathbb{C}[x,y]$.

Algebraic geometers came up with a way around this... but Grothendieck came along & found a better solution.

He defined a new kind of space called an affine scheme such that the correspondence between algebra & geometry is perfect.

We're going to make up a category AffSch whose objects are "affine schemes" & morphisms are maps between them, in such a way that $\text{AffSch}^{op} = \text{CommRing}$. Just take $\text{AffSch} = \text{CommRing}^{op}$.

Example: the circle is an affine scheme, namely the commutative ring $\mathbb{Z}[x,y]/\langle x^2+y^2-1\rangle$. The real plane is the affine scheme $\mathbb{Z}[x,y]$.

"The circle is included in the plane" means we have a homomorphism of commutative rings $\mathbb{Z}[x,y] \to \mathbb{Z}[x,y]/\langle x^2+y^2-1\rangle$.

In "noncommutative geometry", we come up with a new kind of space so that $? = \text{Ring}^{op}$.