Duality

Every category $C$ has an opposite $C^{op}$. $C^{op}$ has the same objects as $C$, but the morphisms are "turned around". So there's a 1-1 correspondence between morphisms in $C$ & morphisms in $C^{op}$, with $f: x \to y$ in $C$ corresponding to a morphism $f^{op}: y \to x$ in $C^{op}$. We compose morphisms in $C^{op}$ by: $f^{op} \circ g^{op} = (g \circ f)^{op}$.

The study of how categories $C$ relate to their partners $C^{op}$ is called duality.

Note: $(C^{op})^{op} = C$. Just like for finite-dimensional vector spaces, $(V^*)^* \cong V$ via a natural isomorphism.

It turns out that THE DUAL OF GEOMETRY IS ALGEBRA.

In geometry we study "points"; in algebra we study addition & multiplication. Descartes realized we can reduce (a lot of) geometry to algebra through "analytic geometry."

We can associate to any finite dimensional vector space over $\mathbb{R}$ a commutative ring $O(V)$ consisting of all polynomial functions on $V$. If $V = \mathbb{R}^n$, the algebra $O(V)$ consists of polynomials in the coordinate functions $x_1, \ldots, x_n : O(V) = \mathbb{R}[x_1, \ldots, x_n]$.

So we go from a "space" $V$ (a bunch of points) to an algebra $O(V)$. Then we can describe certain subspaces $X$ of $V$ as quotients $O(V) : O(V) \twoheadrightarrow O(X) = O(V)/I$ for an ideal $I$.

Example: the unit circle is a subspace of the plane: $S' \subset \mathbb{R}^2$, where $S' = \{(x, y) : x^2 + y^2 - 1 = 0\}$. Then there's an algebra $O(S')$ of polynomials on $S'$ with $O(S') = \mathbb{R}[x, y]/\langle x^2 + y^2 - 1 \rangle$. So the 1-1 map $S' \to \mathbb{R}^2$ gets turned around, giving $O(\mathbb{R}^2) \to O(S')$ which is just restriction $f \mapsto f|_{S'}$. Moreover $f, g \in O(\mathbb{R}^2)$ restrict to the same function on $S'$ iff $f - g \in \langle x^2 + y^2 - 1 \rangle$.

Algebraic geometry is the study of geometry using commutative rings.
Our idea is: subspaces of $\mathbb{V}$ should correspond to quotient rings of $\mathcal{O}(\mathbb{V})$, or ideals of $\mathcal{O}(\mathbb{V})$.

Problems:

1) What about $\langle x^2+y^2+1 \rangle \subseteq \mathcal{O}(\mathbb{R}^2)$?

The function $x^2+y^2+1$ doesn't vanish on $\mathbb{R}^2$, so it seems the corresponding subspace of $\mathbb{R}^2$ is $\emptyset$.

But there's another, simpler ideal corresponding to $\emptyset \subset \mathbb{R}^2$, namely $\langle 1 \rangle$.

So $\emptyset = \{ (x,y) : 1 = 0 \}$, but we're getting 2 ideals corresponding to the same subspace. One way out is to use $\mathbb{C}$ instead of $\mathbb{R}$.

2) Alas, using $\mathbb{C}$ doesn't completely fix the problem of 2 different ideals corresponding to the same subspace.

There's a complex line in $\mathbb{C}^2$ given by $x = y$, with ideal $\langle x-y \rangle \subseteq \mathbb{C}[x,y]$.

But $(x-y)^2$ also vanishes only on this line, so we're getting a different ideal defining the same subspace: $\langle (x-y)^2 \rangle \subseteq \mathbb{C}[x,y]$.

Algebraic geometers came up with a way around this... but Grothendieck came along & found a better solution.

He defined a new kind of space called an affine scheme such that the correspondence between algebra & geometry is perfect.

We're going to make up a category $\text{AffSch}$ whose objects are "affine schemes" & morphisms are maps between them, in such a way that $\text{AffSch}^{op} = \text{CommRing}$. Just take $\text{AffSch} = \text{CommRing}^{op}$.

Example: the circle is an affine scheme, namely the commutative ring $\mathbb{Z}[x,y]/\langle x^2+y^2-1 \rangle$. The real plane is the affine scheme $\mathbb{Z}[x,y]$.

"The circle is included in the plane" means we have a homomorphism of commutative rings $\mathbb{Z}[x,y] \rightarrow \mathbb{Z}[x,y]/\langle x^2+y^2-1 \rangle$.

In "noncommutative geometry", we come up with a new kind of space so that $? = \text{Ring}^{op}$.