Geometry
Algebraic geometry:
\[ C = \text{affine schemes} \]

Topology
\[ C = \text{compact Hausdorff spaces} \]

Set theory
\[ C = \text{sets} \]

Commutative Algebra
Ring Theory
\[ C^\text{op} = \text{commutative rings} \]

C*-algebra Theory
\[ C^\ast = \text{commutative C*-algebras} \]

Logic
\[ C^\ast = \text{atomic Boolean algebras} \]

Look at \( \text{C}^{\text{Haus}} = \text{compact Hausdorff space, continuous maps} \)

From a compact Hausdorff space \( X \) on algebra
\[ C(X) = \{ f : X \to \mathbb{C} : f \text{ is continuous} \} \]

This is an algebra with
\[
(f + g)(x) = f(x) + g(x) \\
(fg)(x) = f(x)g(x) \\
(cf)(x) = cf(x) \quad c \in \mathbb{C}
\]

It's a commutative algebra.

It's a \( \ast \)-algebra with \( (f^\ast)(x) = \overline{f(x)} \),

meaning an algebra \( A \) with \( \ast : A \to A \) s.t.
\[
(f + g)^\ast = f^\ast + g^\ast \\
(fg)^\ast = g^\ast f^\ast \\
(cf)^\ast = \overline{c} f^\ast
\]

Also, \( C(X) \) has a norm \( \|f\| = \sup_{x \in X} |f(x)| \) This makes sense since \( X \) is compact.

This makes \( C(X) \) into a \( C^\ast \)-algebra, meaning that:
\[
\|fg\| \leq \|f\| \|g\| \\
\|f^\ast\| = \|f\| \\
\|f\| \|f^\ast\| = \|f\|^2 \\
\|f^\ast f\| = \sup_{x \in X} (f^\ast f)(x) \\
= \sup_{x \in X} |f(x)|^2 \\
= (\sup_{x \in X} |f(x)|)^2 \\
= \|f\|^2
\]

So \( C(X) \) is a commutative \( C^\ast \)-alg.
Next: can we take a morphism \( \varphi: X \to Y \) in \( \text{C} \text{Haus} \), that is a cont. map, and turn it into a (homo) morphism of comm. \( C^* \)-algebras.

A homomorphism between \( C^* \)-algs., say \( F: A \to B \), is a map s.t.
- \( F(a+b) = F(a) + F(b) \) \( a, b \in A \)
- \( F(ab) = F(a)F(b) \)
- \( F(ca) = cF(a) \) \( c \in \mathbb{C} \)
- \( F(a^*) = F(a)^* \)
- \( \exists K > 0 \) s.t. \( \|F(a)\| < K \|a\| \) \( \forall a \in A \)

All these imply \( \|F(a)\| = \|a\| \)

So we get a category

\[ \text{Comm} \quad C^* \text{Alg} = \left\{ \text{comm. } C^* \text{-algebras, } C^* \text{-algebra homomorphisms} \right\} \]

How does a cont. map \( \varphi: X \to Y \) between compact Hausdorff spaces give a \( C^* \)-alg. homo. between \( C(X) \) and \( C(Y) \)?

We'll get one

\( \varphi^*: C(Y) \to C(X) \)

by:
- \( \varphi^*(f)(x) = f(\varphi(x)) \) \( f \in C(Y), \ x \in X \)
- or:
- \( \varphi^*(f) = f \circ \varphi \)

This is why algebra is the "dual" of geometry - it goes backwards:

\( \varphi: X \to Y \) gives \( \varphi^*: C(Y) \to C(X) \)

Also \( (\varphi \circ \varphi)^* = \varphi^* \circ \varphi^* \) (check this)
So we’re getting a functor:

\[ C : \text{Chaus} \rightarrow \text{Comm } C^* \text{-Alg}^{op} \]
\[ x \mapsto C(x) \]
\[ \mathcal{I} : x \mapsto x \quad \mathcal{I}^* : C(Y) \rightarrow C(X) \]

C* -alg

Gelfand–Naimark Thm: This functor is an equivalence of categories.

I.e. there’s a functor going back:

\[ \text{Spec} : \text{Comm } C^* \text{-Alg}^{op} \rightarrow \text{Chaus} \]

s.t. \( \text{Spec} \circ C \cong \mathcal{I} \) and \( \text{Co Spec} \cong \mathcal{I}^* \text{Comm } C^* \text{-Alg} \)

\[ \text{natural isomorphism} \]

What’s Spec?

Given a comm. \( C^* \)-alg. \( A \), how do we get a space \( \text{Spec}(A) \)?

Let’s do \( A = C(X) \).

Then \( \text{Spec}(C(X)) \) should give back \( X \).

How do we recover the points of \( X \) starting from \( C(X) \)?

What’s a point in \( X \)?

In terms of \( \text{Chaus} \), what’s a point of \( X \)?

It’s a map \( \mathcal{I} : \{ \ast \} \rightarrow X \) where \( \{ \ast \} \) is the one-point space.

\[ \text{c.e. given } x \in X \text{ thus a map} \]
\[ \mathcal{I} : \{ \ast \} \rightarrow X \]
\[ \ast \mapsto x \]

\[ \ast \quad \mathcal{I} \rightarrow \quad x \]

\[ \ast \mapsto x \]

Conversely any map \( \mathcal{I} : \{ \ast \} \rightarrow X \)

determines a point in \( X \).

Our functor \( C : \text{Chaus} \rightarrow \text{Comm } C^* \text{-alg} \) will turn \( \mathcal{I} : \{ \ast \} \rightarrow X \)

into a homomorphism

\[ \mathcal{I}^* : C(X) \rightarrow C(\{ \ast \}) \]
\[ f \mapsto f \circ \mathcal{I} \]

In fact, \( C(\{ \ast \}) \cong C \) where \( g \in C(\{ \ast \}) \) gives \( g(\ast) \in C \)

So we get

\[ \mathcal{I}^* : C(X) \rightarrow C(\{ \ast \}) \]
\[ f \mapsto f \circ \mathcal{I} \]

\[ f \mapsto f \circ \mathcal{I} \quad (\ast) \]

\[ f(\ast) \]
A point $x \in X$ gives a homomorphism $C(X) \rightarrow \mathbb{C}$

$\Phi \rightarrow (x \mapsto f(x))$

In short: any point $x \in X$ gives a homomorphism from $C(X)$ to $\mathbb{C}$ called evaluation at $x$.

**Lemma** Distinct points of $X$ give distinct homomorphisms $C(X) \rightarrow \mathbb{C}$.

(“There are enough continuous functions to separate points” for a compact Hausdorff space)

**Lemma** Any $C^*$-alg. homomorphism $\Psi: C(X) \rightarrow \mathbb{C}$ comes from a point $x \in X$ via: $\Psi(f) = f(x)$ $\forall f \in C(X)$.

So we get a 1-1 correspondence between points $x \in X$ and homomorphisms $\Psi: C(X) \rightarrow \mathbb{C}$.

So given any comm. $C^*$-algebra $A$ we define a set of points $Spec(A) = \{ \Psi: A \rightarrow \mathbb{C}; \Psi \text{ is a } C^* \text{-alg. homomorphism} \}$

There's a topology making $Spec(A)$ into a compact Hausdorff space.

In this topology $\Psi_e$ converges to $\Psi$ iff $\Psi_e(a) \rightarrow \Psi(a)$ for all $a \in A$.

Finally, given a $C^*$-alg. hom. $F: A \rightarrow B$, how do we get a map of spaces $Spec(F): Spec(B) \rightarrow Spec(A)$?

$Spec(F)(\Psi)(a) = \Psi(F(a))$ $\forall \Psi: B \rightarrow \mathbb{C}$ $C^*$-alg homs.

$a \in A$

$F(a) \in B$

$\Psi(F(a)) \in \mathbb{C}$
So we get functors $\mathcal{C}$

$\mathcal{C}$: Haus $\xrightarrow{\text{Spec}}$ Comm. $C^*$ Algebra $\xrightarrow{\text{motion}}$ which are inverses (up to not. iso.)

Note: injections on the space $\mathcal{S}$ $\xrightarrow{\text{spectrum}}$ Comm Rings $\xrightarrow{\text{motion}}$ Homomorphism from a comm. ring to a field

Point in a space $\xrightarrow{\text{motion}}$ Quotient ring or ideal (surjection or epimorphism $R \rightarrow S$)

Subspace $\xrightarrow{\text{incl}}$ (inclusion $Y \rightarrow X$)

or monomorphism