GEOMETRY	ALGEBRA (COMMUTATIVE)
Algebraic geometry: C = [affine schemes]	Ring theory: C^{op} = [commutative rings]
Topology: C = [compact Hausdorff spaces]	C^*-algebra theory: C^{op} = [commutative C^*-algebras]
Set theory: C = [sets]	Logic: C^{op} = [atomic Boolean algebras]

Look at CHaus = [compact Hausdorff spaces, continuous maps].

From a compact Hausdorff space X, we get a commutative algebra $C(X) = \{f: X \to \mathbb{C} : f \text{ is continuous}\}$. It's a $*$-algebra with $(f^*)(x) = \overline{f(x)}$, meaning an algebra A with $*: A \to A$ s.t. $(f+g)^* = f^* + g^*$, $(fg)^* = g^* f^*$, $(cf)^* = \bar{c} f^*$. Also, $C(X)$ has a norm $\|f\| = \sup_{x \in X} |f(x)|$, which makes sense by compactness. This makes $C(X)$ into a C^*-algebra, meaning: $\|fg\| \leq \|f\| \|g\|$, $\|f^*\| = \|f\|$, & $\|f^* f\| = \|f\|^2$. So $C(X)$ is a commutative C^*-algebra.

Next, can we turn a morphism $\phi: X \to Y$ in CHaus into a morphism of comm. C^*-algs? A homomorphism $F: A \to B$ between C^*-algs. is a map s.t. $F(a+b) = F(a) + F(b)$, $F(ab) = F(a)F(b)$, $F(ca) = cF(a)$, $F(a^*) = F(a)^*$, $\|F(a)\| \leq K\|a\|$ for some $K > 0$. All these imply $\|F(a)\| = \|a\|$.

So we get a category CommC*Alg = [comm. C^*-algs., C^*-alg. homomorphisms].

How does a continuous map $\phi: X \to Y$ b/w compact Hausdorff spaces give a C^*-alg. homomorphism $C(X)$ & $C(Y)$? We'll get one, $\phi^*: C(Y) \to C(X)$ defined by $\phi^*(f) = f \circ \phi$, the "pull back" of f along ϕ.

Note $(\phi \circ \psi)^* = \psi^* \circ \phi^*$. So we're getting a functor

$$\begin{aligned} \mathrm{CHaus} &\xrightarrow{C} \mathrm{CommC^*Alg}^{op} \\ X &\longmapsto C(X) \\ \phi: X \to Y &\longmapsto \phi^*: C(Y) \to C(X) \end{aligned}$$

Gelfand-Naimark Thm: This functor is an equivalence of categories, i.e. there's a functor Spec: $\mathrm{Comm}\,C^*\mathrm{Alg}^{op} \to \mathrm{CHaus}$ such that $\mathrm{Spec} \circ C \simeq 1_{\mathrm{CHaus}}$ & $C \circ \mathrm{Spec} \simeq 1_{\mathrm{Comm}C^*\mathrm{Alg}^{op}}$ are natural isomorphisms.

What is Spec? Given a comm. C^*-alg. A, how do we get $\mathrm{Spec}(A)$? Let's do $A = C(X)$. Then $\mathrm{Spec}(C(X))$ should be X. How do we recover the points of X starting from $C(X)$? But what's a point in X in terms of CHaus? It's a map $\phi: \{*\} \to X$ where $\{*\}$ is the one-point space, which is an object in CHaus. So given $x \in X$, $\phi(*) = x$, & conversely any map $\phi: \{*\} \to X$ determines a point in X. Our functor $C: \mathrm{CHaus} \to \mathrm{Comm.}C^*\mathrm{Alg}^{op}$ will turn $\phi: \{*\} \to X$ into a homomorphism $\phi^*: C(X) \to C(\{*\})$. In fact, $C(\{*\}) \cong \mathbb{C}$ where $g \in C(\{*\})$ gives $g(*) \in \mathbb{C}$. So $\phi^*: C(X) \to \mathbb{C}$ is $\phi^*(f) = f \circ \phi(*) = f(x)$. Thus a point $x \in X$ gives a homomorphism $C(X) \to \mathbb{C}$, $f \mapsto f(x)$, which is just "evaluation at x".

Lemma: Distinct points of X give distinct homomorphisms $C(X) \to \mathbb{C}$. (There are enough continuous functions to separate points, for a compact Hausdorff space).

Lemma: Any C^*-alg. homomorphism $C(X) \xrightarrow{\psi} \mathbb{C}$ comes from some $x \in X$ via $\psi(f) = f(x)$.

Together these lemmas yield a 1-1 correspondence between points $x \in X$ & homos. $\psi: C(X) \to \mathbb{C}$. So given any comm. C^*-alg. A, define $\mathrm{Spec}(A) = \{\psi : A \to \mathbb{C} \mid \psi \text{ is a } \mathbb{C}^*\text{-alg. hom.}\}$.

There's a topology making $\mathrm{Spec}(A)$ into a compact Hausdorff space. In this topology, ψ_i converges to ψ iff $\psi_i(a)$ converges to $\psi(a)$ for all $a \in A$.

Finally, given a C^*-alg. hom. $F: A \to B$, we define a map of spaces $\mathrm{Spec}(F): \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ by $\mathrm{Spec}(F)(\psi) = \psi \circ F$.

So we get functors $\mathrm{CHaus} \underset{\mathrm{Spec}}{\overset{C}{\rightleftarrows}} \mathrm{Comm}\, C^*\mathrm{Alg}^{op}$ which are inverses up to natural isomorphism.

Note:

$$\text{Spaces} \underset{\text{spectrum}}{\overset{\text{functions on the space}}{\rightleftarrows}} \text{Comm. Rings}$$

$$\text{points in a space} \rightleftarrows \text{homomorphisms from a comm. ring to a field}$$

$$\text{subspace} \rightleftarrows \text{quotient ring or ideal}$$