$\chi_S(x) = \begin{cases} 0 & \text{if } x \notin S \\ 1 & \text{if } x \in S \end{cases}$ & conversely any such function gives a subset of X. So 2^X is just another way to think of the power set of X.

The operations $\cup, \cap, {}^c$ on subsets of X correspond to operations $\vee, \wedge, \neg$ on functions $f: X \to \{0,1\}$. We have:

$\chi_{S \cup T} = \chi_S \vee \chi_T$ where $\chi_S \vee \chi_T(x) = \chi_S(x) \vee \chi_T(x)$, & so on.

$(2^S, \vee, \wedge, 0, 1)$ will be a Boolean algebra

Also: $S \subseteq T \Leftrightarrow \chi_S \le \chi_T$ (i.e. $\chi_S(x) = 1 \Rightarrow \chi_T(x) = 1$). Note "$\le$" is not a separate concept, so actually we can say

$\chi_S \le \chi_T \Leftrightarrow \chi_S \wedge \chi_T = \chi_S \Leftrightarrow \chi_S \vee \chi_T = \chi_T$.

Defn A partially ordered set $(A, \le)$ is called a lattice if every pair $a, b \in A$ has a least upper bound $a \vee b$ & a greatest lower bound $a \wedge b$, also a least element $0 = F$ & a greatest element $1 = T$.

Defn A distributive lattice is one where $\wedge$ & $\vee$ distribute over each other.

Defn A Boolean algebra is a distributive lattice A where every $x \in A$ has a complement $\neg x$ such that $x \wedge \neg x = F$ & $x \vee \neg x = T$. (If a complement exists, it's unique).

Ex For any set S, 2^S is a Boolean algebra with pointwise defined $\le$, meaning that given $f, g \in 2^S$, we say $f \le g$ whenever $f(x) \le g(x)$ $\forall x \in S$. It thus has pointwise defined $\wedge, \vee, 0, 1, \& \neg$, e.g. $(\neg f)(x) = \neg f(x)$.

Alas, not every Boolean algebra is isomorphic to one of this form! The Boolean algebras of the form 2^S are "complete atomic Boolean algebras".

Defn A complete Boolean algebra A is one where every subset $S \subseteq A$ has a l.u.b. $\bigvee_{x \in S} x$ & a g.l.b. $\bigwedge_{x \in S} x$ such that they distribute over each other.

Defn An atom in a Boolean algebra A is an element $x \in A$ such that $x \ne 0$ & if $y < x$ then $y = 0$.

Ex In 2^S, the atoms are the elements of S, or singletons $\{s\} \subseteq S$.

Defn A Boolean algebra is atomic if $\forall x \in A$, $x = \bigvee_{\lambda \in \Lambda} y_\lambda$ where $y_\lambda \in A$ are atoms.

There's a category CABA of complete atomic Boolean algebras whose morphisms $\phi : A \to B$ preserve $\vee, \wedge, 0, 1, \neg, \bigvee, \bigwedge$.

There's of course a category Set of sets & functions.

Thm Set is equivalent to $CABA^{op}$ via these functors

$$\begin{aligned} \mathrm{Set} &\longrightarrow CABA^{op} \\ S &\longmapsto 2^S = \mathrm{Hom}_{\mathrm{Set}}(S, 2) \end{aligned} \qquad \& \qquad \begin{aligned} CABA^{op} &\longrightarrow \mathrm{Set} \\ A &\longmapsto \mathrm{Hom}_{CABA}(A, 2) \end{aligned}$$

where morphisms are given via pullback in the obvious way.

Ex $\{f \in L^\infty[0,1] : f(x) = 0 \text{ or } 1 \ \forall x \in [0,1]\}$ is a complete but not atomic Boolean algebra under pointwise operations.