10/19/15

Geometry	Algebra
Algebraic Geometry (affine schemes)	Commutative Rings
Topology (compact Hausdorff spaces)	Commutative C^*-algebras
Set Theory	Complete atomic Boolean algebras (Proposition Logic)
?	Boolean algebras
?	Linear algebra (finite-dim'l vector spaces)
?	Finite abelian groups

The opposite of the category of <u>all</u> Boolean algebras is the category of <u>Stone spaces</u>: compact Hausdorff spaces that are <u>totally disconnected</u>: every open set is closed (& vice versa)

The Boolean algebra of a Stone space X consists of its open subsets, with $A \cup B$ as "$\vee$"
$A \cap B$ as "$\wedge$"
A^c as "$\neg$"

Let FinVect be the category of finite-dimensional vector spaces over favorite field (e.g. $\mathbb{R}$) & linear maps.

What's FinVect^{op}?

A typical morphism in FinVect is $T: \mathbb{R}^n \to \mathbb{R}^m$.

A morphism in FinVect^{op} is thus $T^{op}: \mathbb{R}^m \to \mathbb{R}^n$ .....

suspiciously similar to the transpose $T^t: \mathbb{R}^m \to \mathbb{R}^n$ in FinVect.

In fact, we have an equivalence $\text{FinVect} \xrightarrow{\sim} \text{FinVect}^{op}$, with

$$T: \mathbb{R}^n \to \mathbb{R}^m \longmapsto T^t: \mathbb{R}^m \to \mathbb{R}^n \text{ in FinVect}$$

in FinVect, i.e. $(T^t)^{op}: \mathbb{R}^n \to \mathbb{R}^m$ in FinVect^{op}

We can also get the equivalence $\text{FinVect} \cong \text{FinVect}^{op}$ using $\mathbb{R} \in \text{FinVect}$ as our dualizing object:

$$\text{FinVect} \longrightarrow \text{FinVect}^{op}$$
$$V \longmapsto \hom(V, \mathbb{R}) = V^*$$
$$T: V \to W \longmapsto T^*: W^* \longrightarrow V^* \text{ in FinVect}$$
$$(T^*)^{op}: V^* \longrightarrow W^* \text{ in FinVect}^{op}$$

So, FinVect straddles the worlds of geometry & algebra, being its own opposite.

Also, the category [finite abelian groups, group homomorphisms] is its own "op".

Galois Theory

Galois theory is secretly about dualities between posets.

Def A poset is a partially ordered set $(S, \le)$ where $\le$ is reflexive, transitive, and antisymmetric: $x \le y$ & $y \le x \Rightarrow x = y$.

If $(S, \le)$ is a poset, we get a category with elements of S as objects & there exists a unique morphism $f: x \to y$ iff $x \le y$ $(x, y \in S)$, and no morphisms $f: x \to y$ otherwise.

In fact, the categories we get this way are precisely those with:

1) at most 1 morphism from any object x to any object y
2) if there are morphisms $f: x \to y$ & $g: y \to x$, then $x = y$.

So to a category theorist, a poset is a category with these 2 properties.

Given categories of this kind, a functor is really just an order preserving map $f: (S, \le) \to (T, \le)$, i.e. a function s.t. $x \le y$ in $S \Rightarrow f(x) \le f(y)$ in T.

Given a category of this sort coming from the poset $(S, \le)$, its opposite comes from the poset $(S, \le^{op})$ where $x \le^{op} y$ iff $y \le x$.
We'll write $x \ge y$ for $x \le^{op} y$.

What are adjoint functors between categories of this sort?

Def Given categories C, D, we say a functor $L: C \to D$ is the left adjoint of a functor $R: D \to C$, or R is the right adjoint of L, if there is a natural 1-1 correspondence

$$\hom_D(Lx, y) \cong \hom_C(x, Ry) \qquad \forall x \in C,\ y \in D$$

$\boxed{\text{Ex}}$ Let $L: \text{Set} \to \text{Grp}$ send any set S to the free group on S L = liberty! = freedom
and $R: \text{Grp} \to \text{Set}$ send any group G to its underlying set.
Here $\hom_{\text{Grp}}(LS, G) \cong \hom_{\text{Set}}(S, RG)$

$\boxed{\text{Ex}}$ What are adjoint functors between posets $(S, \leq)$ & $(T, \leq)$?
It's a pair of order-preserving functions

$$\begin{tikzcd} (S, \leq) \arrow[r, bend left, "L"] & (T, \leq) \arrow[l, bend left, "R"] \end{tikzcd}$$

such that: $Lx \leq y \iff x \leq Ry$
This comes from $\hom_D(Lx, y) \cong \hom_C(x, Ry)$

$\boxed{\text{Def}}$ A pair of adjoint functors between posets is called a <u>Galois correspondence</u>.

$\boxed{\text{Thm}}$ Suppose $(S, \leq) \underset{R}{\overset{L}{\rightleftarrows}} (T, \leq)$ is a Galois correspondence. Then we get an order-preserving map $RL: (S, \leq) \to (S, \leq)$.
Let's write $\bar{x}$ for RLx.
Then $x \leq \bar{x} \quad \forall\, x \in S$ $(Lx \leq Lx \Rightarrow x \leq RLx)$
and $\bar{\bar{x}} = \bar{x} \quad \forall\, x \in S$
So we say $\bar{\ }$ is a closure operator on the poset $(S, \leq)$.
Similarly write y° for LRy.
Then $y^\circ \leq y \quad \forall\, y \in T$
and $(y^\circ)^\circ = y^\circ \quad \forall\, y \in T$
So $\circ$ behaves like the "interior" operation on subsets of a top. space — it's a closure operator on $(T, \leq)^{op}$.
Finally, L & R give a bijection between <u>closed</u> elements of S (meaning $x \in S$ w/ $\bar{x} = x$) & <u>open</u> elements of T (meaning $y \in T$ s.t. $y^\circ = y$).