Groupoids

Def. - A morphism \( f: x \to y \) in a category has an inverse \( g: y \to x \) if \( fg = 1_y \) & \( gf = 1_x \). If \( f \) has an inverse, it's unique, so we write it as \( f^{-1} \). A morphism with an inverse is called an isomorphism. If there's an isomorphism \( f: x \to y \), we say \( x \) & \( y \) are isomorphic.

Def. - A groupoid is a category where all morphisms are isomorphisms.

Example - Any group \( G \) gives a groupoid with one object, \( * \), & morphisms \( g: * \to * \) corresponding to elements \( g \in G \), with composition coming from multiplication in \( G \). Conversely, any 1-object groupoid gives a group. So a group is a 1-object groupoid. More generally, if \( C \) is any category & \( x \in C \), the isomorphisms \( f: x \to x \) form a group under composition, called the automorphism group \( \text{Aut}(x) \).

Example - Given any category \( C \), there's a groupoid, the core \( C_0 \) of \( C \), whose objects are those of \( C \) & whose morphisms are the isomorphisms of \( C \), composed as before.

Example - If \( \text{FinSet} = [\text{finite sets, functions}] \), then \( \text{FinSet}_0 = [\text{finite sets, bijections}] \). And if \( n \) is your favorite \( n \)-element set, then \( \text{Aut}(n) = S_n \), the symmetric group. So \( \text{FinSet}_0 \) "unifies" all the symmetric groups.

Example - Suppose \( G \) is a group acting on a set \( X: \lambda: G \times X \to X, (g,x) \to gx \). Often people form the set \( X/G \), the quotient set where an elt. \([x]\) is an equivalence class of elts \( x \in X \) where \( x \sim y \) iff \( y = gx \) for some \( g \in G \). But a "better" thing is to form the translation groupoid \( X//G \), where: objects are elements \( x \in X \), & a morphism from \( x \) to \( y \) is a pair \((g,x)\) where \( g \in G \) & \( gx = y \); \( x \xrightarrow{(g,x)} y \). The composite of \( x \xrightarrow{(g,x)} y \) & \( y \xrightarrow{(h,z)} z \) is \( x \xrightarrow{(hg,x)} z \).

In \( X/G \) we say \( x \) & \( y \) are "equal" if \( gx = y \); in \( X//G \) we say they are isomorphic, or more precisely, we have a chosen isomorphism \((g,x): x \to y\).
To a first approximation, a "moduli space" is a set $X/G$, given some obvious topology, while a "moduli stack" is a groupoid $X//G$, where the set of objects & morphisms have topologies.

Example - Let $X$ be the set of line segments in the Euclidean plane. Let $G$ be the Euclidean group of the plane: all bijections $g: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that preserve distances. $G$ acts on $X$:

More precisely, $X = \mathbb{R}^2 \times \mathbb{R}^2$ & $G$ acts on it via $g(p,q) = (gp, gq)$. We're not counting $(p,q)$ the same as $(q,p)$. We're allowing $p = q$.

$X/G$ is the "moduli space of line segments": $X/G \cong [0, \infty)$.

Given a line segment $(p,q)$, there's a line segment $(q,p)$, but $(p,q) \sim (q,p)$, so they have the same equivalence class & hence give the same element of $X/G$.

Next consider $X//G$. Now objects are line segments & morphisms are like: $(p,q) \xrightarrow{(g, p, q)} (p', q')$ where $gp = p'$ & $gq = q'$ as in the picture.

Given a groupoid $C$, we can form:

1) the set $C$ of isomorphism classes of objects $[x]$ where $[x] = [y]$ iff $x \cong y$.

2) for any $[x] \in C$, a group $\text{Aut}(x)$ where $x$ is any representative of $[x]$. Note if $x \cong y$ then $\text{Aut}(x) \cong \text{Aut}(y)$ as groups.

Thm.- Given a groupoid $C$, we can recover $C$ up to equivalence from $C$ & all the groups $\text{Aut}(x)$, one for each isomorphism class in $C$.

Example - $C = \text{FinSet}_0$. $C \cong \mathbb{N}$. & for each $n \in \mathbb{N}$ we get a group $S_n$ which we've seen is isomorphic to $\text{Aut}(x)$ for any $x \in \text{FinSet}_0$ with $n$ elements.

Example - $C = X//G$ where $X$ is the set of line segments & $G$ is the Euclidean group of the plane. $C \cong [0, \infty)$.

In general, $X//G = X/G$ because both are names for the set of equivalence classes $[x]$ where $x \sim y$ iff $y = gx$ for some $g \in G$. 
But $X//G$ has more information, namely all the automorphism groups $\text{Aut}(x)$, one for each equivalence class. So in our example, what's $\text{Aut}((p,q))$? It's $\mathbb{Z}_2$ if $p \neq q$ since there's a reflection preserving $(p,q)$. If $p=q$, it's the group $O(2)$ of all orthogonal $2 \times 2$ matrices, i.e. all rotations & reflections fixing $p \in \mathbb{R}^2$. 