But $X/\!/G$ has more information, namely all the automorphism groups $\mathrm{Aut}(x)$, one for each equivalence class. So in our example, what's $\mathrm{Aut}((p,q))$? It's $\mathbb{Z}_2$ if $p \neq q$ since there's a reflection preserving (p,q). If $p=q$, it's the group $O(2)$ of all orthogonal 2×2 matrices, i.e. all rotations & reflections fixing $p \in \mathbb{R}^2$.

Moduli Spaces & Moduli Stacks

Given a groupoid C, let $\underline{C}$ be the set of isomorphism classes of objects. Often $\underline{C}$ will have the structure of a space (e.g. topological space, manifold, algebraic variety, scheme, etc.) Then $\underline{C}$ is called a <u>moduli space</u>.

Example: If G is a group acting on a set X, we get a groupoid $X/\!/G$, the <u>translation</u> groupoid, where:

- objects are elements of X
- morphisms

$$x \xrightarrow{(g,x)} y$$

are pairs (g,x) with $g \in G$, $x \in X$, & $y = gx$.

Then $\underline{X/\!/G} \cong X/G$ where X/G has elements $[x]$ with $x \sim y$ iff $y = gx$ for some $g \in G$.

Recall:

Thm: The groupoid $X/\!/G$ is equivalent to the groupoid with:

- one object $[x]$ for each $[x] \in X/G$
- one morphism $f: [x] \to [x]$ for each morphism $f: x \to x$ where x is any chosen representative of the equivalence class $[x]$.

Note: if $[x] \neq [y]$, there are no morphisms between them.

We often call X/G a <u>moduli space</u>, & $X/\!/G$ the <u>moduli stack</u>.

Last time we looked at an example:

Example: "The moduli stack of line segments" in Euclidean geometry. Here,

$X = \mathbb{R}^2 \times \mathbb{R}^2 \ni (p,q)$

$G = O(2) \ltimes \mathbb{R}^2$ is the Euclidean group of the plane

Here we think of (p,q) as a line segment with a chosen 1st & 2nd endpoint, which can be equal.

Then the moduli space is $X/G \cong [0,\infty)$, the space of lengths.

$$[(p,q)] \mapsto |p-q|$$

The moduli stack $X/\!/G$ keeps track of symmetries:

$$\mathrm{Aut}\,[(p,q)] \cong \mathrm{Aut}\,((p,q))$$

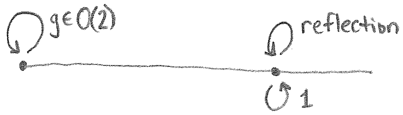
is the subgroup of G consisting of all $g \in G$ with $(gp, gq) = (p,q)$.

$\mathrm{Aut}((p,q)) \cong \mathbb{Z}/2$ if $p \neq q$

$\mathrm{Aut}((p,q)) \cong O(2)$ if $p = q$

$\cong SO(2) \rtimes \mathbb{Z}/2$

So the moduli stack looks like:



Example: "The moduli space of triangles"

Let G be the Euclidean group as before, but now let X be the set of triangles: $X = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$. These are triangles with named vertices that can be equal.

The moduli space X/G is the set of isomorphism classes of triangles.

Now $X/G \cong [0,\infty)^3$

$$[(p,q,r)] \mapsto (|p-q|, |q-r|, |r-p|)$$

Here it seems that if p,q,r are all distinct, then (p,q,r) has as automorphisms only the identity. If we define a triangle to be an unordered triple of points in $\mathbb{R}^2$, then an equilateral triangle would have S_3 as automorphisms, & isosceles would have $S_2 \cong \mathbb{Z}/2$. This gives a more interesting moduli stack.

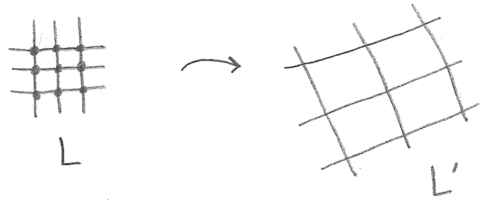
Example: A Riemann surface is a 2-dim. smooth manifold with charts $\phi_i : U_i \to \mathbb{C}$ such that $\phi_i \circ \phi_j^{-1}$ is analytic (=holomorphic). Every Riemann surface that's homeomorphic to the plane is isomorphic (as a Riemann surface) to $\mathbb{C}$. Every Riemann surface homeomorphic to the sphere is isomorphic to the <u>Riemann sphere</u> $\mathbb{C}P^1 \cong \mathbb{C} \cup \{0\}$.

There are lots of nonisomorphic ways to make a torus into a Riemann surface – these are <u>elliptic curves</u>. Every elliptic curve is isomorphic to one of this form:

take a lattice $L \subseteq \mathbb{C}$, i.e. a subgroup of $(\mathbb{C}, +, 0)$ that's isomorphic to $\mathbb{Z}^2$, & form $\mathbb{C}/L$, getting a torus with obvious charts $\phi_i : U_i \to \mathbb{C}$, & thus an elliptic curve.

When do two lattices L, L' give isomorphic elliptic curves $\mathbb{C}/L \cong \mathbb{C}/L'$?

Answer: Iff $L' = \alpha L$ for some nonzero $\alpha \in \mathbb{C}$.



There's a groupoid $\underline{C}$ with:

- elliptic curves as objects
- isomorphisms of Riemann surfaces as morphisms

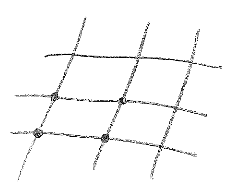
and we're seeing $\underline{C} \simeq X/\!/G$ where X is the set of lattices & $G = \mathbb{C}^*$ (nonzero complex numbers under multiplication).

So X/G is called the moduli space of elliptic curves, &

$X/\!/G$ is the moduli stack of elliptic curves.

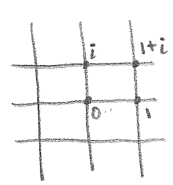
There are two elliptic curves with a bigger automorphism group:

typical elliptic curve



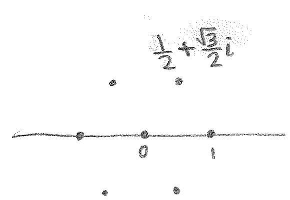
has $\mathbb{Z}/2$ as automorphisms ($180°$ rotation, $(-1)^2 = 1$)

Gaussian elliptic curve



has $\mathbb{Z}/4$ as automorphisms ($i^4 = 1$)

Eisenstein elliptic curve



has $\mathbb{Z}/6$ as automorphisms