Klein Geometry

Def: A homogeneous $G$-space for some group $G$ is a $G$-set $X$, i.e. a set $X$ with a map

$$G \times X \rightarrow X$$

$$(g, x) \mapsto gx$$

Such that $g, (gx) = (g, g_2)(x)$, $1x = x$

which is transitive, i.e.

$$\forall x, y \in X \exists g \in G, gx = y$$

Example: In Euclidean plane geometry, $G = O(2) \times \mathbb{R}^2$ is the Euclidean group and $X = \mathbb{R}^2$ is the Euclidean plane with $g = (r, t) \in O(2) \times \mathbb{R}^2$ acting on $x \in \mathbb{R}^2$ by: $gx = rx + t$.

In non-Euclidean geometry, the parallel postulate fails. Here, the Euclidean group is replaced by some other 3-dimensional Lie group.

Example: In spherical geometry, $G = O(3)$, i.e.

$$G = \{ g : \mathbb{R}^3 \rightarrow \mathbb{R}^3 : g \text{ is linear} \land gx \cdot gy = x \cdot y \text{ for all } x, y \in \mathbb{R}^3 \}$$

$X = S^2 = \{ x \in \mathbb{R}^3 : x \cdot x = 1 \}$

Here we can define a set of lines, namely great circles, but any two distinct lines intersect in two points, so parallel postulate fails, though other axioms of Euclid hold.
Example: In hyperbolic geometry we let $\mathbb{R}^{2,1}$ be $\mathbb{R}^3$ with the dot product $(x, y, z) \cdot (x', y', z') = xx' + yy' - zz'$ and let $G = O(2,1) = \{ g: \mathbb{R}^{2,1} \rightarrow \mathbb{R}^{2,1} : g \text{ is linear} \}$ $g \cdot x = x' \quad \forall x, y \in \mathbb{R}^{2,1}$

and $X = H^2 = \{ x \in \mathbb{R}^{2,1} : x \cdot x = -1 \}$ $= \{ (x, y, z) : x^2 + y^2 - z^2 = -1 \}$ the hyperboloid.

As in spherical geometry, we can define a line to be an intersection of $X$ with some plane through the origin.

The parallel postulate fails because for any line $l$ and any point $p$ not on that line, there are infinitely many lines (e.g. $l'$ and $l''$) containing $p$ but parallel to $l$ (i.e. not intersecting $l$).

In projective (plane) geometry every pair of distinct lines intersects in exactly one point!

Points in projective geometry are really lines through the origin (your eye).
In projective geometry $X = \mathbb{P}^2$ consists of lines through the origin in $\mathbb{R}^2$.

and $G = GL(3, \mathbb{R}) = \{ g: \mathbb{R}^3 \to \mathbb{R}^3 : g \text{ linear, invertible} \}$

... or, since transformations $x \mapsto ax \ (0 \neq a \in \mathbb{R})$ act trivially on $X$, we can use the projective general linear group

$$G = PGL(3, \mathbb{R}) = GL(3, \mathbb{R})/\{ xI : x \in \mathbb{R}^3 \}$$

which is an 8-dimensional group, as opposed to the previous groups, which were 3-dimensional.

$\mathbb{P}^2$ can be identified with $S^2/\sim$ where

$x \sim y \text{ iff } y = \pm x,$

or therefore with $D^2/\sim$ where

$x \sim y \text{ if } x$ & $y$ are the boundary of the disc $D^2$ and they're diametrically opposite.

parallel train tracks
So $\mathbb{RP}^2$ can be seen as $\mathbb{R}^2$ (homeomorphic to the interior of $\mathbb{D}^2$) together with "points at infinity" coming from the boundary of the disc. We can define a line in $\mathbb{RP}^2$ to be a plane through the origin in $\mathbb{R}^3$, which contains lots of points in $\mathbb{RP}^2$ (which were lines through the origin).

Any pair of distinct lines intersect in a unique point, and any pair of distinct points lie on a unique line.

Indeed, in projective plane geometry, any theorem has a "dual" version where the role of points and lines are switched. This is a special case of duality for posets, with $p \preceq l$ meaning $p$ lies on $l$.

Klein noticed that in all the kinds of geometry mentioned so far, we actually have two homogeneous $G$-spaces: the set of points $X$, but also the set of lines $Y$. We also are interested in other homogeneous $G$-spaces, e.g. in 3d geometry we'd have a set of planes; or in 2d projective geometry we have the set of flags:

i.e. point-line pairs, where the point lies on the line, etc.
So Klein's idea was:

a geometry is simply a group, and a type of figure (point, line, flag, triangle, etc.) is a homogeneous G-space \( X \), with an element \( x \in X \) being a figure of that type.

We can keep track of (classify) all these homogeneous G-spaces by:

Thm: Suppose \( X \) is a homogeneous G-space. Pick an element \( x \in X \) and let \( H \leq G \) be the stabilizer of \( x \), i.e., the subgroup

\[
H = \{g \in G : gx = x\},
\]

Then let \( G/H \) be the set of equivalence classes \([g]\) where \( g \equiv g' \iff g' = gh \). Then \( G/H \) is a homogeneous G-space with:

\[
g[g'] = [gg']
\]

and as G-spaces we have

\[
X \cong G/H \text{ via } \alpha : gx \mapsto [g] \quad \forall g \in G
\]

with the obvious inverse.

Here \( \alpha \) is a map of G-spaces: \( \alpha(gx) = g\alpha(x) \).

This allowed Klein to redefine a type of figure to be simply a subgroup of \( G \), since a subgroup \( H \leq G \) gives a transitive G-space \( G/H \), and every transitive G-space is isomorphic to one of those.