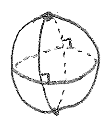
Klein Geometry

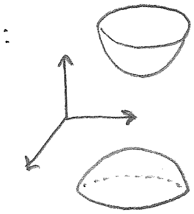
Def. – A <u>homogeneous</u> <u>G-space</u> for some group G is a set X on which G acts transitively, i.e. there's a map $G \times X \to X$ such that
$$(g, x) \mapsto gx$$
$g_1(g_2 x) = (g_1 g_2)x$, $1x = x$, & $\forall\, x, y \in X\ \exists\, g \in G$ w/ $gx = y$.

Ex. – In Euclidean plane geometry, $G = O(2) \ltimes \mathbb{R}^2$ is the <u>Euclidean group</u> & $X = \mathbb{R}^2$ is the <u>Euclidean plane</u> with $g = (r, t) \in G$ acting on $x \in X$ by $gx = rx + t$.

In non-Euclidean geometry, the parallel postulate fails. Here the Euclidean group is replaced by some other 3-dimensional Lie group.

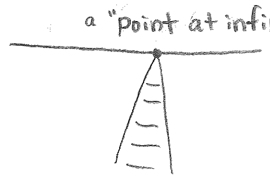
Ex. – In spherical geometry, $G = O(3) = \{g : \mathbb{R}^3 \to \mathbb{R}^3 \mid g$ is linear & $gx \cdot gy = x \cdot y\ \forall\, x, y \in \mathbb{R}^3\}$, & $X = S^2 = \{x \in \mathbb{R}^3 \mid x \cdot x = 1\}$. Here we can define a set of <u>lines</u>, namely great circles:  But any 2 distinct lines intersect at 2 points, so the parallel postulate fails, yet other axioms of Euclidean geometry hold.

Ex. – In hyperbolic geometry, we let $\mathbb{R}^{2,1}$ be $\mathbb{R}^3$ with the dot product $(x, y, z) \cdot (x', y', z') = xx' + yy' - zz'$, let $G = O(2,1) = \{g : \mathbb{R}^{2,1} \to \mathbb{R}^{2,1} \mid g$ is linear & $gx \cdot gy = x \cdot y\ \forall\, x, y \in \mathbb{R}^{2,1}\}$, & let $X = H^2 = \{x \in \mathbb{R}^{2,1} \mid x \cdot x = -1\}$ $= \{(x, y, z) \mid x^2 + y^2 - z^2 = 1\}$ be the hyperboloid:

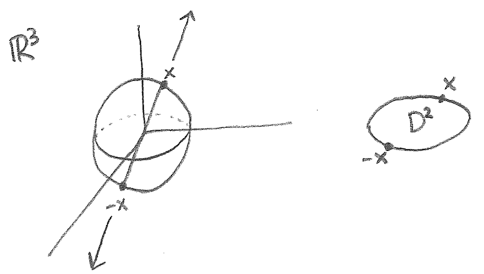


As in spherical geometry, we can define a <u>line</u> to be an intersection of X with some plane through the origin. The parallel postulate fails because for any line ℓ & any point P not on ℓ, there are infinitely many lines ℓ' containing P yet not intersecting ℓ.

In projective (plane) geometry, every pair of distinct lines intersect in exactly 1 point!

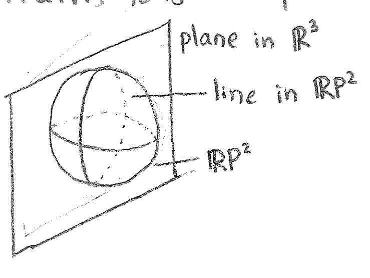


In projective geometry, $X = \mathbb{R}P^2 = \{\text{lines through the origin in } \mathbb{R}^3\}$, & $G = GL(3,\mathbb{R}) = \{g : \mathbb{R}^3 \to \mathbb{R}^3 \mid g \text{ linear \& invertible}\}$ or, since transformations $x \mapsto \alpha x$ for some $0 \neq \alpha \in \mathbb{R}$ act trivially on X, we can use the <u>projective</u> <u>general</u> <u>linear</u> group $G = PGL(3,\mathbb{R}) = GL(3,\mathbb{R}) / \{\alpha I : \alpha \in \mathbb{R}\}$, which is an 8-dimensional group, as opposed to the previous groups, which were 3-dimensional.



$\mathbb{R}P^2$ can be identified with $S^2/\sim$ where $x \sim y$ iff $y = \pm x$ or, therefore with $D^2/\sim$ where $x \sim y$ iff x & y are on the boundary of the disc D^2 & diametrically opposite. So $\mathbb{R}P^2$ can be seen as $\mathbb{R}^2$ (homeomorphic to the interior of D^2) together with "points at infinity" coming from the boundary of the disc.

We can define a <u>line</u> in $\mathbb{R}P^2$ to be a plane through the origin in $\mathbb{R}^3$, which contains lots of points in $\mathbb{R}P^2$ (which were lines through the origin).



Any pair of distinct lines intersect in a unique point, & any pair of distinct points lie on a unique line. Indeed in projective plane geometry, any theorem has a "dual" version where the role of points & lines are switched.

This is a special case of duality for posets, with $p < \ell$ meaning p lies on ℓ.

Klein noticed that in all the kinds of geometry mentioned so far, we have two homogeneous G-spaces: the set X of points but also the set Y of lines. We also are interested in other homogeneous G-spaces, e.g. in 3d geometry we'd have a set of planes, or in 2d Euclidean geometry we have the set of flags, i.e. point-line pairs where the point lies on the line, etc.

So Klein's idea was: a <u>geometry</u> is simply a group, & a <u>type of figure</u> (point, line, flag, triangle, etc.) is a homogeneous G-space X, whose elements $x \in X$ are figures of that type.

We can classify all these homogeneous G-spaces by:

Thm.- Suppose X is a homogeneous G-space. Pick an element $x \in X$. Let $H \subseteq G$ be the stabilizer of x, i.e. the subgroup $H = \{g \in G : gx = x\}$. Then let G/H be the set of equivalence classes $[g]$ where $g \sim g'$ iff $g' = gh$. Then G/H is a homogeneous G-space with $g[g'] = [gg']$. & $X \cong G/H$ as G-spaces via $gx \overset{\alpha}{\mapsto} [g]$ $\forall g \in G$. (Here α is a map of G-spaces, meaning $\alpha(gx) = g\alpha(x)$. $\forall g \in G$.

This allowed Klein to redefine a <u>type of figure</u> to be simply a subgroup of G, since a subgroup $H \subseteq G$ gives a transitive G-space G/H, & every transitive G-space is isomorphic to one of those.