11/23/15

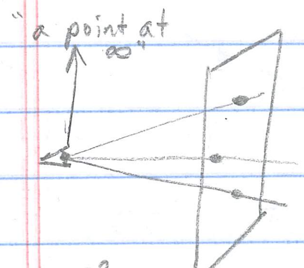
Klein Geometry

We've seen that:

- a <u>geometry</u> is a group G
- a <u>type of figure</u> in this geometry is a subgroup $H \subseteq G$
- the set of figures of that type is G/H: a homogeneous G-space.

How can we do geometry this way?
We need G-invariant relations between figures.

Example: projective plane geometry: $G = PGL(3, \mathbb{R})$



$\mathbb{R}^2 \subseteq \mathbb{R}P^2$

$X = \{\text{lines through the origin}\}$

X is a homogeneous G-space, so $X \cong G/H$ where $H \subseteq G$ is the stabilizer of your favorite point $p \in X$:

$$H = \{h \in G : hp = p\}$$

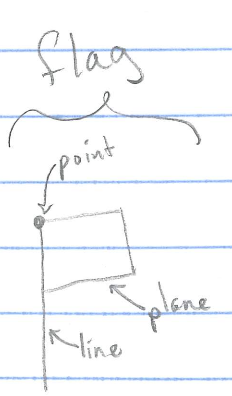
An invariant relation between points is a relation, i.e. a subset $R \subseteq X \times X$ such that

$$(p,q) \in R \Rightarrow (gp, gq) \in R \text{ for all } p, q \in X,\ g \in G.$$

But the only invariant relations in this example are $p = q$ and $p \neq q$ (because distance is not preserved by G).

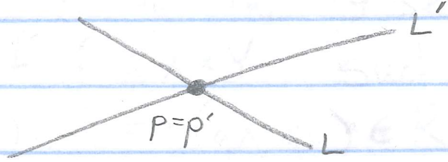
More interestingly, let $Y = \{A \subseteq B : A \text{ is a 1d subspace of } \mathbb{R}^3 \text{ and } B \text{ is a 2d subspace of } \mathbb{R}^3\}$

or $Y = \{\text{flags}\}$, where a flag is a point $p \in \mathbb{R}P^2$ lying on a line $L \subseteq \mathbb{R}P^2$



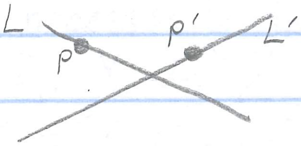
G acts transitively on Y (Even the Euclidean group does), and there are various invariant relations between flags, i.e. subsets $R \subseteq Y \times Y$ invariant under G.

For example: (p, L) and (p', L'), where $p = p'$, $L \neq L'$



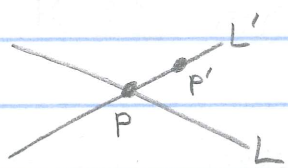
Dually, $p \neq p'$, $L = L'$:

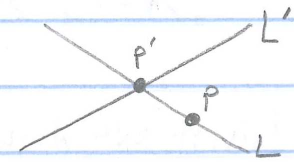
Also, $p \neq p'$, $L \neq L'$, $p \notin L'$, $p' \notin L$



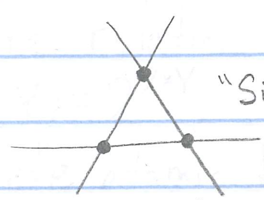
$p = p'$, $L = L'$

$p \neq p'$, $L \neq L'$, $p \in L'$ — dually $p \neq p'$, $L \neq L'$, $p' \in L$





All six of these relations are visible here:



"Six flags"

Pick one flag,
then pick the second flag from the six. Each choice of second flag has one of these six relations with the first flag.

For any group G, we can make up a category $G\mathrm{Rel}$ where:

- objects are G-sets
- morphisms are invariant relations,

Where an <u>invariant relation</u> $R: X \nrightarrow Y$ from the G-set X to the G-set Y is a relation, i.e. a subset $R \subseteq X \times Y$ such that:

$$(x,y) \in R \Rightarrow (gx, gy) \in R \quad \forall x \in X, y \in Y, g \in G.$$

How do we compose morphisms?
Given any relations $R: X \nrightarrow Y$ and $S: Y \nrightarrow Z$ (not necessarily invariant) we can compose them to get $S \circ R: X \nrightarrow Z$:

$$S \circ R = \{(x,z) \in X \times Z : \exists y \in Y \text{ such that } (x,y) \in R \text{ and } (y,z) \in S\}$$

If R & S are invariant, so is $S \circ R$.

There is a category Rel ($G\mathrm{Rel}$ with the trivial group G) where:

- objects are sets
- morphisms are relation

Here $\hom(X,Y) = 2^{X \times Y}$. Recall, for any set S, 2^S is a complete atomic boolean algebra (CABA), with

$\subseteq$ as $\leq$

$\cap$ as $\wedge$ $(=\mathrm{glb})$

$\cup$ as $\vee$ $(=\mathrm{lub})$

^{c} as $\neg$

So in Rel, $\hom(X,Y)$ is not merely a set, it's a CABA. The same is true for GRel: e.g. if $R: X \nrightarrow Y$, $S: X \nrightarrow Y$ are invariant, so are $R \cap S$, $R \cup S$, R^c. (Need to check more stuff...)

In fact, Rel and GRel are "CABA-enriched categories". What's an enriched category?
In category theory we want to overthrow the tyranny of sets: instead of working in Set all the time, we try to prove results that hold in many categories. But the very definition of category uses sets:

"A category is a ~~set~~ class of objects, and for each pair of objects x, y a set $\hom(x,y)$, and a composition function:

$$\circ : \hom(x,y) \times \hom(y,z) \to \hom(x,z)$$

etc..."

The idea in enriched category theory is to generalize, replacing Set by some other category V and say
A V-enriched category is a class of objects, and for each pair of objects x, y an object $\hom(x,y) \in V$, and a composition morphism in V:

$$\circ : \hom(x,y) \otimes \hom(y,z) \to \hom(x,z) \quad \text{etc...}$$

Here we need V to be a "monoidal category", i.e. a category with some sort of "tensor product" $\otimes$.

It turns out that CABAs form a monoidal category, so it makes sense to talk about a CABA-enriched category, and Rel & GRel are such.