11/23/15

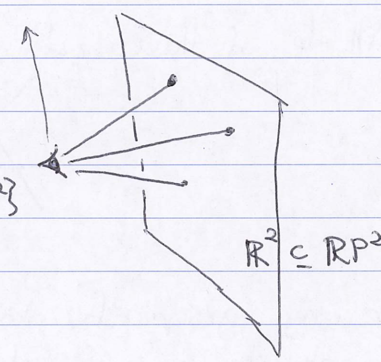
Klein Geometry

We've seen that:

- a <u>geometry</u> is a group G
- a <u>type of figure</u> in this geometry is a subgroup $H \subseteq G$
- the set of figures of that type is G/H: a homogeneous G-space

How can we do geometry this way?

We need G-invariant relations between figures.



Ex projective plane geometry:

$G = PGL(3,\mathbb{R})$

$X = \{\text{lines through the origin in } \mathbb{R}^3\} = \{\text{pts in } \mathbb{RP}^2\}$

X is a homogeneous G-space, so

$X \cong G/H$ where $H \subseteq G$ is the stabilizer of your favorite point $p \in X$:

$H = \{h \in G : hp = p\}$

An invariant relation between points is a relation, i.e. a subset $R \subseteq X \times X$ s.t. $(p,q) \in R \Rightarrow (gp, gq) \in R$ for all $p,q \in X$, $g \in G$

But the only invariant relations in this example are

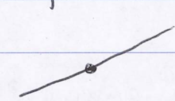
$p = q$ and $p \neq q$

because distance is not preserved by G.

More interestingly, let $Y = \{A \subseteq B : A$ is a 1-dim. subspace of $\mathbb{R}^3$ & B is a 2-dim. subspace of $\mathbb{R}^3\}$

$= \{\text{flags}\}$

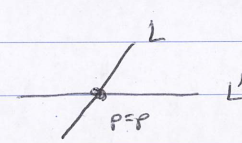
where a flag is a point $p \in \mathbb{RP}^2$ lying on a line $L \subseteq \mathbb{RP}^2$



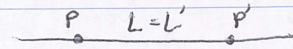
G acts transitively on Y (even the Euclidean group does), and there are various invariant relations between flags, i.e. subsets $R \subseteq Y \times Y$ invariant under G.

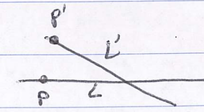
For example:

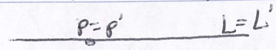
One invariant relation between (p,L) & (p',L') says "$p = p'$ and $L \neq L'$"

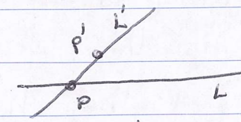


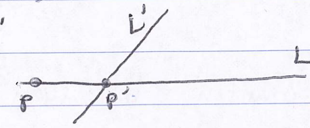
(continued)

Or: "$L = L'$ and $p \neq p'$" 

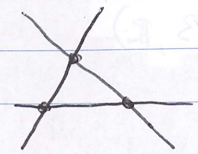
Or: "$p \neq p'$ and $L \neq L'$" 

Or: "$p = p'$ and $L = L'$" 

Or: "$p \in L'$ but $L \neq L'$ and $p \neq p'$" 

Or: "$p' \in L$ but $L \neq L'$ and $p \neq p'$" 

All 6 of these relations are visible here:



"6 flags"

For any group G, we can make up a category $G\mathrm{Rel}$ where:
- objects are G-sets
- morphisms are invariant relations

where an invariant relation $R: X \nrightarrow Y$ from the G-set to the G-set Y is a relation, i.e. a subset $R \subseteq X \times Y$ such that:

$$(x,y) \in R \Rightarrow (gx, gy) \in R \quad \forall\, x \in X,\ y \in Y,\ g \in G$$

How do we compose morphisms?
Given any relations $R: X \nrightarrow Y$ and $S: Y \nrightarrow Z$ (not nec. invariant)
we can compose them to get $S \circ R: X \nrightarrow Z$:

$$S \circ R = \{(x,z) \in X \times Z : \exists\, y \in Y \text{ s.t. } (x,y) \in R \text{ and } (y,z) \in S\}$$

If R & S are invariant so is $S \circ R$.

There's a category Rel where
- objects are sets
- morphisms are relations

Here

$$\hom(X,Y) = 2^{X \times Y}$$

Recall: for any set S, 2^S is a CABA: a complete atomic boolean algebra, with

$\subseteq$ as $\le$

$\cap$ as $\wedge$ $(= \text{glb})$

$\cup$ as $\vee$ $(= \text{lub})$

^{c} as $\neg$

So in Rel, $\hom(X,Y)$ is not merely a set, it's a CABA. The same is true for GRel: e.g. if $R: X \nrightarrow Y$, $S: X \nrightarrow Y$ are invariant, so is $R \cap S$, $R \cup S$, R^c

In fact Rel & GRel are "CABA-enriched categories"

What's an enriched category?

In category theory we want to overthrow the tyranny of sets: instead of working in Set all the time, we try to prove results that hold in many categories. But the very definition of category use sets:

A category is a class of objects, and for each pair of objects x, y a set $\hom(x,y)$, and a composition function

$\circ: \hom(x,y) \times \hom(y,z) \longrightarrow \hom(x,z)$

etc...

The idea in enriched cat. theory is to generalize, replacing Set by some other category V and say:

A V-inriched category is a class of objects, and for each pair of x, y an object $\hom(x,y) \in V$, and a composition morphism in V:

$\circ: \hom(x,y) \otimes \hom(y,z) \longrightarrow \hom(x,z)$

etc...

Here we need V to be a "monoidal category", i.e. a category with some sort of "tensor product" $\otimes$.

It turns out that CABA's form a monoidal category, so it makes sense to talk about a CABA-enriched category, & Rel & GRel are such.