Klein Geometry

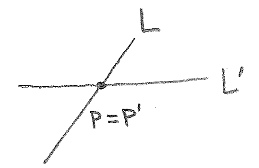
We've seen that:

- a geometry is a group G
- a type of figure in this geometry is a subgroup $H \subseteq G$
- the set of figures of that type is G/H : a homogeneous G-space

How can we do geometry this way?
We need G-invariant relations between figures.

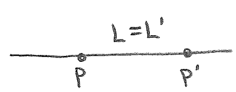
Example: projective plane geometry
$G = PGL(3,\mathbb{R})$, $X = \{\text{lines through the origin in } \mathbb{R}^3\} = \{\text{points in } \mathbb{RP}^2\}$
X is a homogeneous G-space, so $X \cong G/H$ where $H \subseteq G$ is the stabilizer of your favorite point $p \in X$: $H = \{h \in G : hp = p\}$.
An invariant relation between points is a relation, i.e. a subset $R \subseteq X \times X$ such that $(p,q) \in R \Rightarrow (gp, gq) \in R \quad \forall \ p,q \in X \ \& \ g \in G$.
But the only invariant relations in this example are $p = q$ & $p \neq q$ because distance is not preserved by G.
More interestingly, let $Y = \{A \subseteq B : A \text{ is 1-dim. subspace } \& \ B \text{ is 2-dim. subspace of } \mathbb{R}^3\} = \{\text{flags}\}$, where a flag is a point $p \in \mathbb{RP}^2$ lying on a line $L \subseteq \mathbb{RP}^2$.
G acts transitively on Y (even the Euclidean group does), & there are various invariant relations between flags, i.e. subsets $R \subseteq Y \times Y$ invariant under G.

For example:



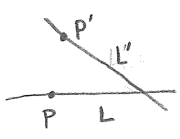
One invariant relation between (p,L) & (p',L') says "$p = p'$ & $L \neq L'$".

Or:



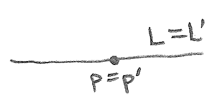
"$L = L'$ & $p \neq p'$"

Or:

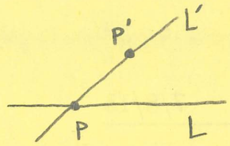


"$p \neq p'$ & $L = L'$ & $p \notin L'$ & $p' \notin L$"

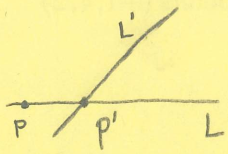
Or:



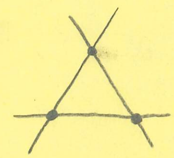
"$p = p'$ & $L = L'$"

Or: 

"$p \in L'$ but $L \neq L'$ & $p \neq p'$"

Finally: 

"$p' \in L$ but $L \neq L'$ & $p \neq p'$"

All 6 of these relations are visible here:  "6 flags"

For any group G, we can make up a category $G\mathrm{Rel}$ where:

- objects are G-sets
- morphisms are invariant relations

where an invariant relation $R: X \nrightarrow Y$ from the G-set X to the G-set Y is a relation, i.e. a subset $R \subseteq X \times Y$ such that $(x,y) \in R \Rightarrow (gx, gy) \in R$ $\forall\ x \in X, y \in Y, g \in G$.

How do we compose morphisms?

Given any relations $R: X \nrightarrow Y$ & $S: Y \nrightarrow Z$ (not necessarily invariant), we can compose them to get $S \circ R: X \nrightarrow Z$:

$$S \circ R = \{(x,z) \in X \times Z : \exists\, y \in Y \text{ s.t. } (x,y) \in R \ \&\ (y,z) \in R\}$$

If R & S are invariant then so is $S \circ R$.

There's a category Rel where:

- objects are sets
- morphisms are relations

Here, $\hom(X,Y) = 2^{X \times Y}$.

Recall for any set S, 2^S is a complete atomic boolean algebra (CABA), with $\subseteq$ as $\leq$, $\cap$ as $\wedge$ ($=$ glb), $\cup$ as $\vee$ ($=$ lub), ^{c} as $\neg$.

So in Rel, $\hom(X,Y)$ is not merely a set, it's a CABA. The same is true for $G\mathrm{Rel}$: e.g. if $R: X \nrightarrow Y$, $S: X \nrightarrow Y$ are invariant, so is $R \cap S$, $R \cup S$, R^c.

In fact, Rel & $G\mathrm{Rel}$ are "CABA-enriched categories."

What's an enriched category?

In category theory, we want to overthrow the tyranny of sets: instead of working in Set all the time, we try to prove results that hold in many categories. But the very definition of category uses sets. The idea in enriched category theory is to generalize, replacing Set by some other category V & say:

A V-enriched category is a class of objects, & for each pair of objects x, y an object $\hom(x,y) \in V$, & a composition morphism $\circ : \hom(x,y) \otimes \hom(y,z) \longrightarrow \hom(x,z)$ in V, etc.

Here we need V to be a "monoidal category", i.e. a category with some sort of "tensor product" $\otimes$.

It turns out that CABA's form a monoidal category, so it makes sense to talk about a CABA-enriched category, & Rel & GRel are such.