Enriched categories & internal monoids

A monoid "is the same as" a 1-object category: if you have a category $\mathcal{C}$ with one object $x$, there's a monoid $\text{hom}(x,x)$ with multiplication

$$o: \text{hom}(x,x) \times \text{hom}(x,x) \rightarrow \text{hom}(x,x).$$

Conversely, given a monoid $M$ you can build a category with one object $x$ and $\text{hom}(x,x) = M$, with composition being multiplication in $M$.

More generally, suppose $V$ is a monoidal category, i.e. a category with tensor product $\otimes: V \times V \rightarrow V$ obeying some rules.

Then recall a $V$-enriched category $\mathcal{C}$ has a class of objects, and for any objects $x, y \in \mathcal{C}$, a "hom-object" $\text{hom}(x,y) \in V$, and composition morphisms:

$$o: \text{hom}(x,y) \otimes \text{hom}(y,z) \rightarrow \text{hom}(x,z)$$

A 1-object $V$-enriched category is the same as a monoid internal to $V$, or monoid in $V$, i.e. an object $M \in V$ with a multiplication $m: M \otimes M \rightarrow M$ that is associative and unital.

Example: Suppose $V = \text{AbGrp}$ with the usual tensor product of abelian groups. Then a monoid in $V$ is called a ring.

It's an abelian group $M$, with a multiplication:

$$m: M \otimes M \rightarrow M$$

an abelian group homomorphism, i.e. a function $m: M \times M \rightarrow M$ that's linear in each argument.
That is, \((a+b)\cdot c = a\cdot c + b\cdot c\)
\[a\cdot (b + c) = a\cdot b + a\cdot c.\]

Example: if \(V = \text{RMod}\) for some commutative ring \(R\), a monoid in \(V\) is called an \(R\)-algebra.

Example: if \(V = \text{Top}\) with usual product \(\times\) of topological spaces as \(\otimes\), a monoid in \(V\) is called a topological monoid. (LIE groups are especially nice versions of these)

Back to our favorite example: Klein geometry.
Let \(G\) be a group, and let \(G\text{-Rel}\) be the category with
- \(G\)-sets as objects
- \(G\)-invariant relations as morphisms.
This is a CABA-enriched category. So if we take one object, i.e. one \(G\)-set \(X\), we can form a 1-object CABA-enriched category with
- \(X\) as the only object
- \(\text{hom}(X, X)\) is the only homset, or "hom-CABA".

Example: Projective plane geometry.
Take \(G = \text{PGL}(3, \mathbb{R})\). \(Y = \{\text{flags}\in \mathbb{P}(L)\} : p \leq L^3\) is a 1d subspace,
\(L \leq \mathbb{R}^3\) is a 2d subspace,
\(p \leq L^2\)
\( \text{hom}(Y, Y) \) is a monoid in CABA. What is it like? Instead of describing all the elements, let's just describe the atoms.

In general, given any group \( G \) and any \( G \)-sets \( X, Y \), what are the atoms in \( \text{hom}(X, Y) \) like? They're invariant relations \( R : X \rightarrow Y \), i.e. \( R \subseteq X \times Y \) such that \( (x, y) \in R \Rightarrow (gx, gy) \in R \). But they're the smallest nonempty subsets of this form. So, any atom \( R \) must contain a point \( (x, y) \), and thus all points of the form \( (gx, gy) \) with \( g \in G \). Indeed, any orbit \( \{(gx, gy) : g \in G\} \subseteq X \times Y \) is an atom in \( \text{hom}(X, Y) \).

So if \( G = \text{PGL}(3, \mathbb{R}) \) and \( Y = \{ \text{flags} \} \), the atoms in \( \text{hom}(Y, Y) \) are the orbits of \( G \) acting on \( Y \times Y \). E.g. the orbit of this

\[
\begin{aligned}
x = (p, L) \\
y = (p', L')
\end{aligned}
\]

is the set of all pairs of flags sharing the same point (and nothing more!)

Last time we saw all six atoms in \( \text{hom}(Y, Y) \):

\[
\begin{aligned}
\end{aligned}
\]
The identity $1$ in $\text{hom}(Y,Y)$ is "two flags are the same".

Note: we can compose invariant relations & $1 \circ 1 = 1$

Let $A$ in $\text{hom}(Y,Y)$ be "having the same point but no more".

$A \circ A = A \cup 1$

If we change the line on a flag twice, the result is either changing the line or getting back to the original flag.

Let $B$ be $\text{hom}(Y,Y)$ be "having the same line but no more" i.e. "changing the point".

$B \circ B = B \cup 1$

$A \circ B$ (change the point, then change the line)

is the atom, "p' \in L but no more".

$B \circ A$ is the atom, "p \in L' but no more".

$A \circ B \circ A = B \circ A \circ B$ is "nothing interesting".

In fact, this is a presentation for our monoid in

$CABA$, $\text{hom}(Y,Y)$.
If we draw A as \(|\)
and B as \(\times\)
then \(A \circ B \circ A = B \circ A \circ B\) is called the "3rd Reidemeister move" or "Yang-Baxter equation":

\[
\begin{array}{c}
\text{This is the only relation in } B_3, \text{ the 3-strand braid group}
\end{array}
\]