11/30/15

Enriched categories & internal monoids

A monoid "is the same as" a 1-object category:
if you have a category C with one object x, there's a monoid $\hom(x,x)$ with multiplication

$$\circ : \hom(x,x) \times \hom(x,x) \to \hom(x,x).$$

Conversely, given a monoid M you can build a category with one object x and $\hom(x,x) = M$, with composition being multiplication in M.

More generally, suppose V is a monoidal category, i.e. a category with tensor product $\otimes : V \times V \to V$ obeying some rules.
Then recall a V-enriched category C has a class of objects, and for any objects $x, y \in C$, a "hom-object" $\hom(x,y) \in V$, and composition morphisms:

$$\circ : \hom(x,y) \otimes \hom(y,z) \to \hom(x,z)$$

A <u>1-object V-enriched category</u> is the same as a <u>monoid internal to V</u>, or <u>monoid in V</u>, i.e. an object $M \in V$ with a multiplication $m : M \otimes M \to M$ that is associative and unital.

Example: Suppose $V = \mathrm{AbGp}$ with the usual tensor product of abelian groups. Then a monoid in V is called a <u>ring</u>.
It's an abelian group M, with a multiplication:

$$m : M \otimes M \to M$$

an abelian group homomorphism, i.e. a function $m : M \times M \to M$ that's linear in each argument.

That is, $(a+b)\cdot c = a\cdot c + b\cdot c$
$$a\cdot(b+c) = a\cdot b + a\cdot c.$$

Example: if $V = R\mathrm{Mod}$ for some commutative ring R, a monoid in V is called an <u>R-algebra</u>.
Example: if $V = \mathrm{Top}$ with usual product $\times$ of topological spaces as $\otimes$, a monoid in V is called a topological monoid. (Lie groups are especially nice versions of these)

Back to our favorite example: Klein geometry.
Let G be a group, and let $G\mathrm{Rel}$ be the category with

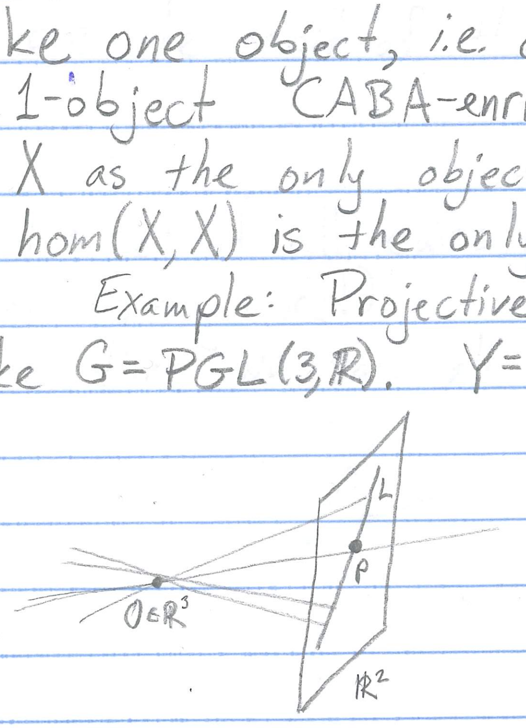
- G-sets as objects
- G-invariant relations as morphisms.

This is a CABA-enriched category. So if we take one object, i.e. one G-set X, we can form a 1-object CABA-enriched category with

- X as the only object
- $\hom(X,X)$ is the only homset, or "hom-CABA".

Example: Projective plane geometry.

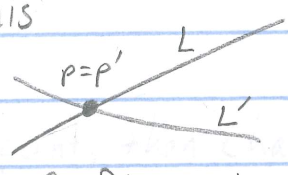
Take $G = PGL(3,\mathbb{R})$. $Y = \{\text{flags}\} = \{(p,L) : p \subseteq \mathbb{R}^3$ is a 1d subspace, $L \subseteq \mathbb{R}^3$ is a 2d subspace, $p \subseteq L\}$



hom(Y,Y) is a monoid in CABA. What is it like?
Instead of describing all the elements, let's just describe the atoms.

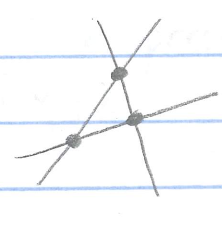
In general, given any group G and any G-sets X, Y, what are the atoms in hom(X,Y) like?
They're invariant relations $R: X \nrightarrow Y$, i.e. $R \subseteq X \times Y$ such that $(x,y) \in R \Rightarrow (gx, gy) \in R$.
But they're the smallest nonempty subsets of this form.
So, any atom R must contain a point (x,y), and thus all points of the form (gx, gy) with $g \in G$. Indeed, any orbit $\{(gx,gy) : g \in G\} \subseteq X \times Y$ is an atom in hom(X,Y).

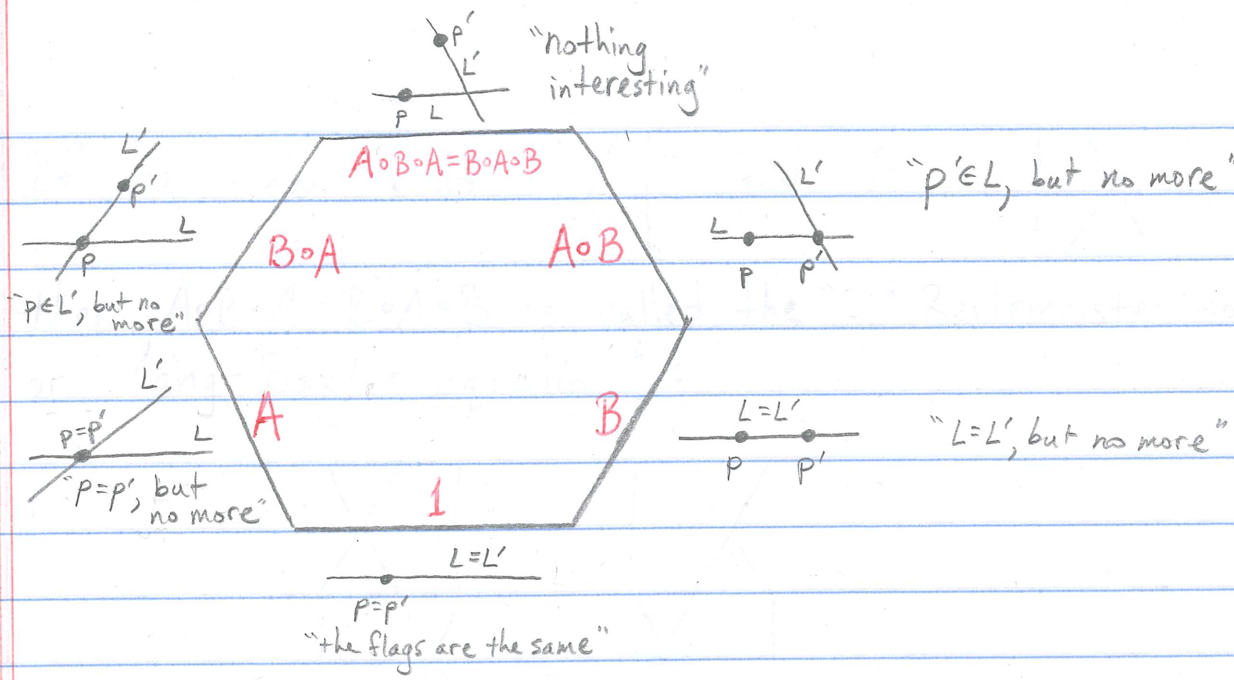
So if $G = PGL(3,\mathbb{R})$ and $Y = \{\text{flags}\}$, the atoms in hom(Y,Y) are the orbits of G acting on $Y \times Y$.
E.g. the orbit of this



$x = (p, L)$
$y = (p', L')$

is the set of all pairs of flags sharing the same point (and nothing more!)
Last time we saw all six atoms in hom(Y,Y):





The identity 1 in hom(Y,Y) is "two flags are the same".

Note: we can compose invariant relations & $1 \circ 1 = 1$

Let A in hom(Y,Y) be "having the same point but no more".

$$A \circ A = A \cup 1$$

If we change the line on a flag twice, the result is either changing the line or getting back to the original flag

Let $B \in$ hom(Y,Y) be "having the same line but no more" i.e. "changing the point".

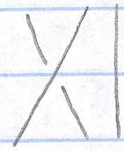
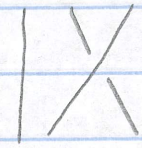
$$B \circ B = B \cup 1$$

$A \circ B$ (change the point, then change the line) is the atom, "$p' \in L$ but no more".

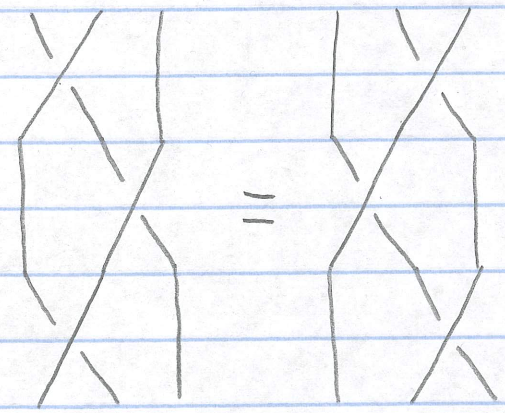
$B \circ A$ is the atom, "$p \in L'$ but no more".

$A \circ B \circ A = B \circ A \circ B$ is "nothing interesting".

In fact this is a presentation for our monoid in CABA, hom(Y,Y)

If we draw A as  and B as 

then $A \circ B \circ A = B \circ A \circ B$ is called the "3rd Reidemeister move" or "Yang-Baxter equation":



This is the only relation in B_3, the 3-strand braid group