Enriched categories & internal monoids

A monoid is "the same" as a 1-object category: if you have a category C with one object x, there's a monoid $\hom(x,x)$ with multiplication $\circ : \hom(x,x) \times \hom(x,x) \to \hom(x,x)$. Conversely given a monoid M you can build a category with one object x & $\hom(x,x) = M$, with composition being multiplication in M.

More generally suppose V is a monoidal category with tensor product $\otimes$. Then recall a V-enriched category C has a class of objects & for any objects $x, y \in C$, a "hom-object" $\hom(x,y) \in V$ & composition of morphisms $\circ : \hom(x,y) \otimes \hom(y,z) \to \hom(x,z)$.

A 1-object V-enriched category is the same as a monoid internal to V, or monoid in V, i.e. an object $M \in V$ with a multiplication $m : M \otimes M \to M$ that's associative & unital.

Examples:

1. Suppose $V = \mathrm{AbGp}$ with $\otimes = \otimes_{\mathbb{Z}}$. Then a monoid in V is called a ring.
2. If $V = R\mathrm{Mod}$ with $\otimes = \otimes_R$, then a monoid in V is called an R-algebra.
3. If $V = \mathrm{Top}$ with $\otimes = \times$, then a monoid in V is a topological monoid.

Back to our favorite example: Klein geometry. Let G be a group, & let $G\mathrm{Rel}$ be the category with:

- G-sets as objects
- G-invariant relations as morphisms

This is a CABA-enriched category. So, if we take one G-set X, we can form a 1-object CABA-enriched category with:

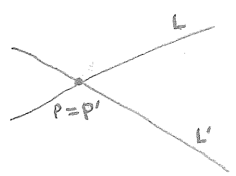
- X as the only object
- $\hom(X,X)$ is the only "hom-CABA"

Example: Projective plane geometry.
Take $G = PGL(3,\mathbb{R})$ & $Y = \{(p,L) : p \subseteq \mathbb{R}^3$ is a 1-dim. subspace, $L \subseteq \mathbb{R}^3$ is a 2-dim. subspace, $p \subseteq L\}$, the set of flags.

$\hom(Y,Y)$ is a monoid in CABA. What's it like? Instead of describing all the elements, let's just describe the atoms.

In general, given any group G & any G-sets X, Y, what are the atoms in $\hom(X,Y)$ like? They're invariant relations $R: X \nrightarrow Y$, i.e. $R \subseteq X \times Y$ such that $(x,y) \in R \Rightarrow (gx, gy) \in R$. But they're the smallest nonempty subsets of this form. So, any atom R must contain a point (x,y) & thus all points of the form (gx, gy) for $g \in G$. Indeed, any orbit $\{(gx,gy) : g \in G\} \subseteq X \times Y$ is an atom in $\hom(X,Y)$.

So, if $G = PGL(3,\mathbb{R})$ & $Y = \{\text{flags}\}$, then the atoms in $\hom(X,Y)$ are the orbits of G acting on $Y \times Y$. E.g. the orbit of this pair of flags

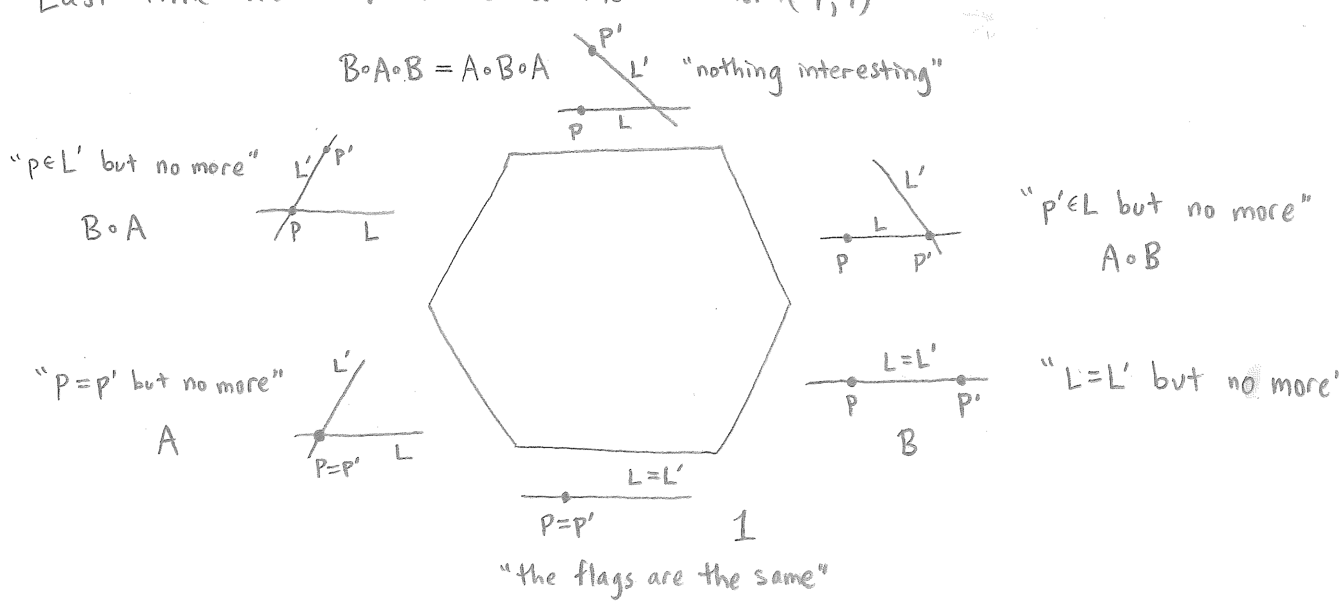


$x = (p, L)$

$y = (p', L')$

is the set of all pairs of flags sharing the same point (but no more!).

Last time we saw all 6 atoms in $\hom(Y,Y)$:

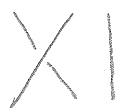


The identity $1 \in \hom(Y,Y)$ is "two flags are the same". Note we can compose invariant relations & $1 \circ 1 = 1$.

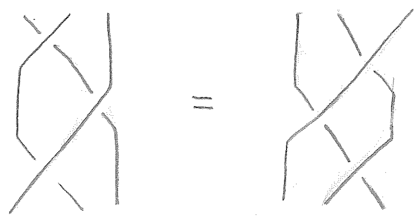
Let $A \in \hom(Y,Y)$ be "having the same point but no more". Then $A \circ A = A \cup 1$, because if you change the line on a flag twice, the result could be changing the line or getting back the original flag.

Let $B \in hom(Y,Y)$ be "having the same line but no more" & "changing the point". Then $B \circ B = B \cup 1$. Now $A \circ B$ is one of our atoms, "$p' \in L$ but no more", while $B \circ A$ is another atom, "$p \in L'$ but no more".

Finally, $A \circ B \circ A = B \circ A \circ B$ is "nothing interesting". In fact this is a presentation for our monoid in CABA, $hom(Y,Y)$.

If we draw A as  & B as , then $A \circ B \circ A = B \circ A \circ B$

is called the "3rd Reidemeister move" or "Yang-Baxter equation":



This is the only relation in B_3, the 3-strand braid group.