LINEAR ALGEBRAIC GROUPS: LECTURE 2

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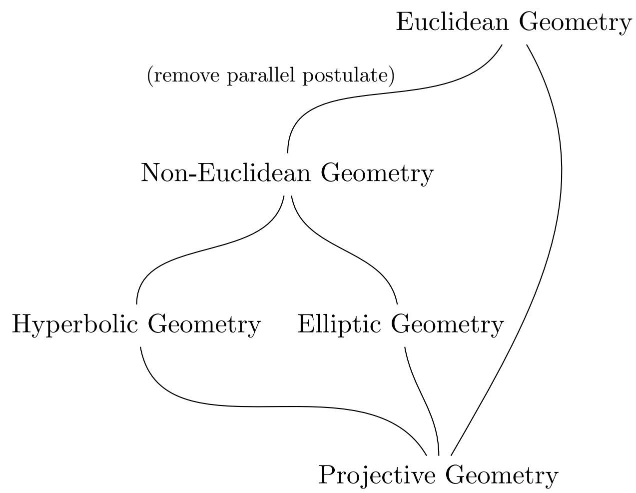
1. Introduction

Having looked at some geometries and their associated symmetry groups, we could consider generalizing the three we have discussed. In particular, we can show that projective geometry, which we will formally define later, in a sense encompasses all of those discussed so far.



In order to show the relationships of the previous geometries, we will build each in nearly the same fashion, for now restricting the construction to the ambient space k^3 for an arbitrary field k.

2. Elliptic Geometry

We consider k^3 with the usual dot product,

$$v \cdot w \;=\; v_1 w_1 + v_2 w_2 + v_3 w_3$$

for all $v, w \in k^3$. Now, consider the sphere

$$X \;=\; \left\{ v \in k^3 : v \cdot c = 1 \right\} \;=\; S^2.$$

We set P to be the set lines through the origin of k^3, so

$$P \;=\; \left\{ p : p \text{ is a 1-dim subspaces of } k^3 \right\}.$$

Similarly, we set L to be the set of planes through the origin, so

$$L \;=\; \left\{ \ell : \ell \text{ is a 2-dim subspaces of } k^3 \right\}.$$

As before, we define an incidence relation, "A point lies on a line", by $p \subseteq \ell$. Now, we can use $SO(3)$ as the symmetry group. Note that it preserves the dot product on k^3, and so it also preserves it on the subspaces contained in P and L. Moreover, it preserves our incidence relation. For any $g \in SO(3)$, $p \in P$ and $\ell \in L$,

$$p \subseteq \ell \quad \Rightarrow \quad gp \subseteq g\ell.$$

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Again, we chose a symmetry group that preserves dot product *and* the incidence relation.

3. Hyperbolic Geometry

We wish to construct hyperbolic geometry in a manner similar to that used for elliptic geometry. To that end, we will choose to let X be a hyperboloid of two sheets, so

$$X \;=\; \left\{(x,y,z)\in k^3 : -x^2-y^2=1\right\}.$$

Now, we would *like* to use the one dimensional and two dimensional subspaces as we did in elliptic geometry, but not all of those will intersect the hyperboloid. Why didn't this matter in the elliptic case? Well, *every* subspace intersects the sphere!

To accommodate this, let

$$P \;=\; \left\{p : p \text{ is a 1-dim subspaces of } k^3\right\}\cap X$$

and

$$L \;=\; \left\{\ell : \ell \text{ is a 2-dim subspaces of } k^3\right\}\cap X.$$

To this collection, we can apply a Lorentzian dot product, defined as

$$v\cdot w \;=\; -v_1w_1 - v_2w_2 + v_3w_3$$

for all $v,w\in k^3$. Utilizing this dot product, we have

$$X \;=\; \left\{v\in k^3 : v\cdot v = 1\right\}.$$

To understand the symmetry group of our geometry, let's first introduce another new group:

$$O(1,2) \;=\; \left\{g\in GL(3) : gv\cdot gw = v\cdot w \;\forall v,w\in k^3\right\},$$

where again we're using our Lorentzian dot product. Now, our symmetry group (which preserves this dot product) becomes $SO(1,2)$, where

$$SO(1,2) \;=\; O(1,2)\cap SL(3).$$

As before, we are choosing to remove reflections. $SO(1,2)$ acts on k^3, preserving our dot product and orientation, so it also preserves them on X, P, and L. As a result, the action of $SO(1,2)$ on our hyperbolic geometry also preserves the incidence relation, so for any $g\in SO(1,2)$, $p\in P$ and $\ell\in L$,

$$p\subseteq \ell \;\Rightarrow\; gp\subseteq g\ell.$$

Note that Einstein realized time-space in $\mathbb{R}^4$ with $O(1,3)$ as a symmetry group.

4. Euclidean Geometry

Recall that in our last lecture, we created non-Euclidean geometry by removing the parallel postulate. Now, let's build Euclidean plane geometry in a way parallel (pun intended) to the above approach for elliptic and hyperbolic geometry. This time, we use a rather strange dot product,

$$v\cdot w \;=\; v_3\cdot w_3$$

for all $v,w\in k^3$. We then have the signs of the three dot products:

	v_1w_1	v_2w_2	v_3w_3
elliptic:	$+$	$+$	$+$
Euclidean:	0	0	$+$
hyperbolic:	$-$	$-$	$+$

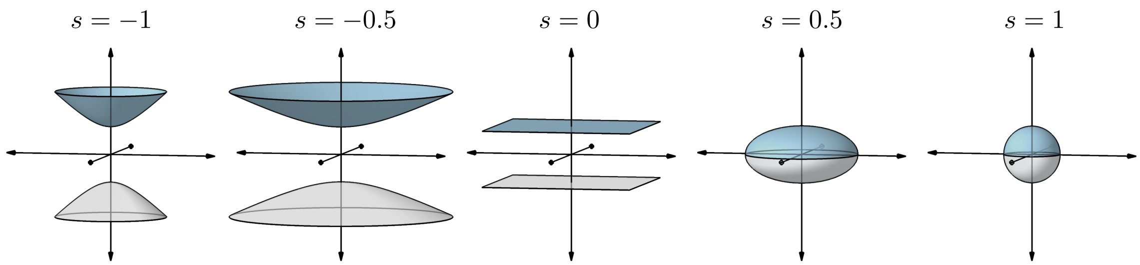
For our intersecting space within k^3, we can choose

$$X \;=\; \left\{v\in k^3 : v\cdot v = v_3v_3 = 1\right\},$$

which is the two parallel planes defined by $v_3=\pm 1$. Each of these surfaces, our X for the three geometries, is defined through the equation

$$X \;=\; s(x^2+y^2)+z^2 = 1,$$

where for Euclidean space $s = 0$. This means Euclidean geometry can be viewed as the limit of the two non-Euclidean cases:



As a symmetry group, it seems anything that would leave the third coordinate unchanged would suffice. However, we wish to use our already defined Euclidean group mentioned in lecture 1:

$$E(2) \;=\; \left\{ g \in GL(3) : g = \left(\begin{array}{c|c} R & v \\ \hline 0\ 0 & 1 \end{array} \right),\ \text{for } R \in SL(2) \text{ and } v \in k^3 \right\}.$$

Note that

$$\left(\begin{array}{c|c} R & v \\ \hline 0\ 0 & 1 \end{array} \right) \begin{pmatrix} x \\ y \\ \pm 1 \end{pmatrix} \;=\; \begin{pmatrix} \alpha \\ \beta \\ \pm 1 \end{pmatrix}$$

for some $\alpha, \beta \in k$, which means that the group maps any element in the planes of X to the same plane. As with the other geometries, the symmetry group preserves both the inner product and the incidence relation.

If you'd like to understand more of why this is the symmetry group, you can read the related blog.

5. Projective Geometry

As already mentioned, all three of the geometries covered can be subsumed into a single one: projective geometry. There is a caveat - we will only worry about the incidence relation, *not* the dot products, so we don't worry about angles and distances. In projective geometry, we still define

$$P \;=\; \left\{ p : p \text{ is a 1-dim subspaces of } k^3 \right\},$$

and

$$L \;=\; \left\{ \ell : \ell \text{ is a 2-dim subspaces of } k^3 \right\}.$$

This was actually the same in all three earlier cases. Likewise, we use the natural incidence relation, where

$$p \mathrm{I} \ell \qquad \Leftrightarrow \qquad p \subseteq \ell.$$

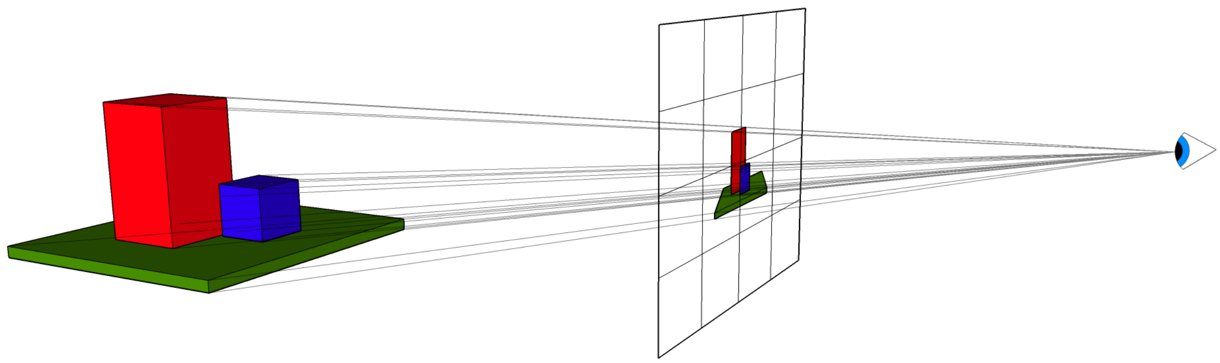
We can take the symmetry group to be $GL(3)$, so that for any $g \in GL(3)$,

$$p \mathrm{I} \ell \qquad \Leftrightarrow \qquad (gp)\, \mathrm{I}\, (g\ell)\,.$$

If we wanted to remove reflections (as we did in the other three), we could choose only those elements in $GL(3)$ with positive determinant. Moreover, there are many elements in $GL(3)$ which have the same effect on our projective space. These lead to what is known as a projective linear group, but we will stick with $GL(3)$ as our symmetry group to keep things simple.

We will denote the projective space of dimension n over a field k as kP^n.

As a bit of history, many people attribute modern approaches to perspective painting (and its associated projective geometry) to Italian Renaissance painter Filippo Brunelleschi, roughly around 1413. His approach could be compared to painting on a piece of glass. Placing a glass between, say, a still life and the painter. Then, points which lie in the same line from the eye are equivalent. This makes the eye the origin of 3-space.



Notice that if we allowed for an infinite plane, instead of a small piece of glass, most lines through the eye would hit the glass - except for those on a plane parallel to the glass containing the eye.

In a similar manner, most one dimensional subspaces of $\mathbb{R}^3$ (points in $\mathbb{R}P^2$) will hit an arbitrary plane *not* containing the origin. Thus, most points $p \in P$ correspond to a point on such a plane. The remaining points of P are called "points at infinity".

In general, we start with an arbitrary field k. Then, we define kP^{n-1} to be the collection of one dimensional subspaces of k^n. Then, most one dimensional subspaces of k^n, intersect the plane

$$X \;=\; \{(x_1, \dots x_{n-1}, 1) : x_i \in k\}$$

in a single point. Thus, we get a 1-1 correspondence between points in X and a big subset of kP^{n-1}. In the case of kP^2, those that do not, which belong to a two dimensional subspace parallel to X, are collapsed down to a single equivalence class, or "point at infinity".

To recap, the examples of plane geometry show that

- Points in any of the three model geometries are points in projective geometry;
- All three symmetry groups are subgroups of $GL(3)$, the symmetry group of projective geometry.