LINEAR ALGEBRAIC GROUPS: LECTURE 3

JOHN SIMANYI

1. Bruhat Cells

Considering an arbitrary field k, recall that

$$kP^n \;:=\; \{\text{1-dimensional subspaces of } k^n\}.$$

Theorem. *As sets, there exists an isomorphism (bijection)*

$$kP^n \;\cong\; k^n + k^{n-1} + \cdots + k^0.$$

These pieces (k^i) are called Bruhat cells, but this is sometimes called the Schubert decomposition of kP^n, since the closures of the Bruhat cells are called Schubert cells.

Proof. Any 1-dimensional subspace of k^{n+1} can be written in the form

$$p \;=\; \langle (x_1, x_2, \ldots, x_{n+1}) \rangle\,,$$

where $(x_1, x_2, \ldots, x_{n+1})$ is not the origin. If $x_{n+1} \neq 0$, we can write any such p as

$$p \;=\; \left\langle \left(\frac{x_1}{x_{n+1}}, \frac{x_2}{x_{n+1}}, \ldots, \frac{x_{n+1}}{x_{n+1}} \right) \right\rangle \;=\; \langle (y_1, y_2, \ldots, , y_n, 1) \rangle\,.$$

There is a clear bijection

$$k^n \;\cong\; \{p \in kP^n : p = \langle (y_1, y_2, \ldots, , y_n, 1) \rangle\}\,.$$

Of course, there are those points in kP^n for which x_{n+1} is zero, and are of the form

$$p \;=\; \langle (x_1, x_2, \ldots, x_n, 0) \rangle\,.$$

If $x_n \neq 0$, we can again divide all coordinates by its value to rewrite such p as

$$p \;=\; \langle (y_1, y_2, \ldots, , y_{n-1}, 1, 0) \rangle\,.$$

This collection is in bijection with k^{n-1}. Via induction, we arrive at our result. □

2. Some Examples

Since $kP^1 = k^1 + k^0$ is just the affine line plus a single point, we generally write it as

$$kP^1 \;=\; k + \{\infty\},$$

which is referred to as "the one point compactification". Using our coordinate approach, we can write

$$k \;\cong\; \left\{p \in kP^1 : p = \langle (x_1, 1) \rangle \right\},$$

and

$$\{\infty\} \;\cong\; \left\{p \in kP^1 : p = \langle (x_1, 0) \rangle \right\}.$$

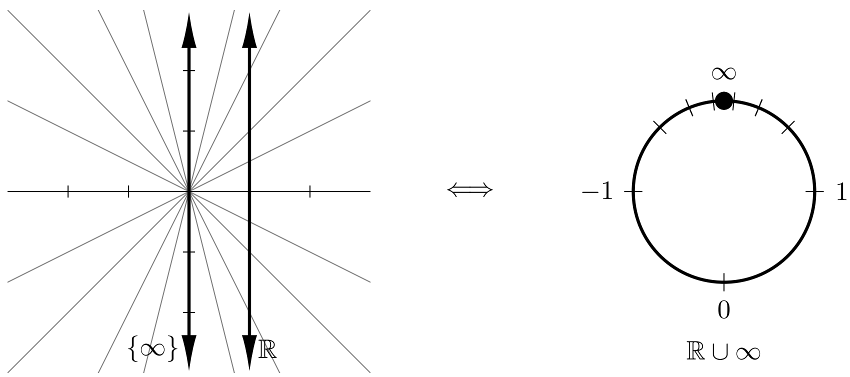
This infinity is precisely the lines of infinite slope in k^2.

Date: September 29, 2016.

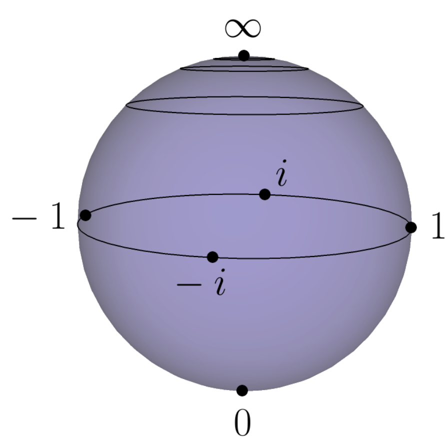
Example 1. If $k = \mathbb{R}$, then

$$\mathbb{R}P^1 \;\cong\; \mathbb{R} + \{\infty\}.$$

This one point compactification has the topology of the circle:



Example 2. If $k = \mathbb{C}$, then $\mathbb{C}P^1 \cong \mathbb{C} + \{\infty\} \cong S^2$, again as a topological space. This can be considered as the Riemann sphere (a simply connected compact Riemann surface), as well as a complex variety.

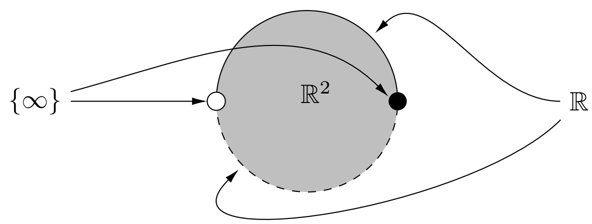


SphereGP.png

Example 3. Returning to $k = \mathbb{R}$, we have that

$$\begin{aligned} \mathbb{R}P^2 &\cong \mathbb{R}^2 + \mathbb{R} + \{\infty\} \\ &\cong \mathbb{R}^2 + \mathbb{R}P^1 \\ &\cong \mathbb{R}^2 + S^1 \\ &\cong S^2/\sim, \end{aligned}$$

again as a topological space. This is <u>*almost*</u> the disk D^2. However, $\mathbb{R}P^2$ cannot be embedded in $\mathbb{R}^3$ without intersection, just like the Klein bottle. Remember that in our original description, we considered points in $\mathbb{R}P^2$ via antipodal identification, so we could look at it as the upper half-sphere (which is essentially a disk), with the added requirement that we identify antipodal points at the equator. As a Bruhat decomposition, we have



3. Projective Geometry in Finite Fields

Any finite field has q elements, where $q = p^n$ for some prime p. Moreover, all fields with q elements are isomorphic, so we write $\mathbb{F}_q$ for "the" field with q elements. Note that

(1) $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$ for a prime p;
(2) For $\mathbb{F}_{p^m}$, $m > 1$, we take $\mathbb{F}_p$ and in a sense "throw in" the roots of some irreducible polynomial with coefficients in $\mathbb{F}_p$.

Question: what is the cardinality of $\mathbb{F}_q P^n$? Well, we can use our Bruhat decomposition to find

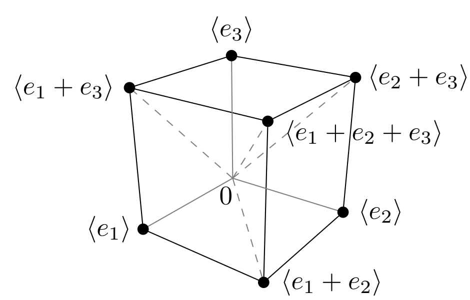
$$\begin{aligned} |kP^n| &= \left|\mathbb{F}_q^n\right| + \left|\mathbb{F}_q^{n-1}\right| + \cdots + \left|\mathbb{F}_q^0\right| \\ &= q^n + q^{n-1} + \cdots + 1 \\ &= \frac{q^{n+1} - 1}{q - 1}. \end{aligned}$$

We call this value, denoted $[n+1]_q$ as the q-integer.

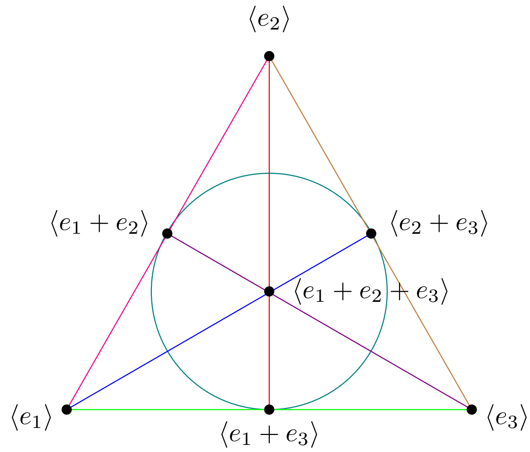
Example 4. We call $\mathbb{F}_2 P^2$ the <u>Fano plane</u>. By our rule,

$$\left|\mathbb{F}_2 P^2\right| = [3]_2 = 2^2 + 2 + 1 = 7.$$

If we view $\mathbb{F}_2^3$ as $\langle e_1, e_2, e_3 \rangle$, similar to our natural basis in $\mathbb{R}^3$, we can picture the seven points as all possible sums of our basis elements:



If we actually draw them in a plane, we can then consider all possible lines (2-dimensional subspaces) that can be created:



Note that any two lines intersect in precisely one point, while any two points lie on precisely one line.

Theorem. *In* _any_ *projective plane* kP^2 :

(1) *Any two distinct points determine a unique line.*
(2) *Any two distinct lines determine a unique point.*

Proof. (1) Given any distinct 1-dimensional subspaces $p, p' \in k^3$, the vector space sum

$$p + p' \;=\; \{v + v' : v \in p, v' \in p'\}$$

is a 2-dimensional subspace, so it determines a (projective) line. By linear algebra, this line is unique.

(2) Given any distinct 2-dimensional subspaces $\ell, \ell' \in k^3$, we claim $\ell \cap \ell'$ is a 1-dimensional subspace, and therefore a (projective) point. Notice that as vector subspaces,

$$\begin{array}{rccccccccc} & \dim(\ell + \ell') & = & \dim(k^3) & = & \dim \ell & + & \dim \ell' & - & \dim(\ell \cap \ell') \\ \Rightarrow & & & 3 & = & 2 & + & 2 & - & 1. \end{array}$$

This (projective) point is again unique by the usual linear algebra. $\square$

Next time we'll try to axiomatize the concept of projective plane. We'll use the two properties we just proved, but that won't quite be enough.