LINEAR ALGEBRAIC GROUPS: LECTURE 5

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1. Projective Geometry from a Kleinian Perspective

According to Klein, a linear algebraic group "gives" us a geometry. Let's first consider $G = GL(n)$. In general, different sets of figures in projective geometry correspond to different subgroups of $GL(n)$.

Definition. Let the Grassmannian $Gr(n, j)$, for $0 \leq j \leq n$, be the set of all j-dimensional subspaces of k^n.

As examples, we have that

$$\begin{aligned} Gr(n,1) &= \left\{\text{points of } kP^{n-1}\right\}, \\ Gr(n,2) &= \left\{\text{lines of } kP^{n-1}\right\}, \\ Gr(n,3) &= \left\{\text{planes of } kP^{n-1}\right\}, \\ Gr(n,j) &= \left\{(j-1)\text{-planes of } kP^{n-1}\right\}, \\ Gr(n,n-1) &= \left\{\text{hyperplanes of } kP^{n-1}\right\}. \end{aligned}$$

Now, $GL(n)$ acts on each Grassmannian, since it acts on k^n mapping subspaces to subspaces of the same dimension. If $L \in GL(n, j)$ and $g \in GL(n)$, then

$$gL = \{gv : v \in L\}.$$

The Grassmannians are all homogeneous $GL(n)$-spaces, which is to say $GL(n)$ acts transitively. Any $L \in Gr(n, j)$ has a basis $\{v_i\}_{i=1}^{j}$, and any other $L' \in Gr(n, j)$ has a basis $\{v_i'\}_{i=1}^{j}$. Basic linear algebra tells us we can find a linear operator $g \in GL(n)$ such that $gv_i = v_i'$ for all i, so there exists a g such that

$$gL = L'.$$

By our "easy" theorem in the last lecture, a Grassmannian is in bijection with a quotient space

$$Gr(n, j) \cong GL(n)/P_{n,j}$$

for linear algebraic subgroup $P_{n,j}$, where $P_{n,j}$ is the subgroup that fixes a chosen $L \in Gr(n, j)$. Subgroups of this form - $P_{n,j}$ - are "maximal parabolic" subgroups of $GL(n)$. Indeed, any linear algebraic group will have some maximal parabolic subgroups that fix the "nicest" types of figures in its associated geometry.

2. Maximal Parabolic Subgroups

To study these $P_{n,j}$ choose a nice $L \in kP^{n-1}$. We can choose

$$L = \{(x_1, x_2, \ldots, x_j, 0, 0, \ldots, 0) \in k^n\},$$

and define

$$P_{n,j} = \{L \in GL(n) : gL = L\}.$$

We begin with a few basic examples. First, we work in k^3.

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Example 1. We can consider

$$P_{3,1} \;=\; \{g \in GL(3) : g \text{ fixes a point in the projective plane.}\}$$

We will use the convention that a star ($\ast$) is a wildcard, which can take any value in k. This means our particular L can be written as $\begin{pmatrix} \ast & 0 & 0 \end{pmatrix}^T$, and by testing the idea we get that the subgroup consists of matrices of the form

$$\begin{pmatrix} \ast & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \end{pmatrix}\begin{pmatrix} \ast \\ 0 \\ 0 \end{pmatrix} \;=\; \begin{pmatrix} \ast \\ 0 \\ 0 \end{pmatrix}.$$

Hence,

$$P_{3,1} \;=\; \{A \in GL(3) : a_{21}, a_{31} = 0\}.$$

Similarly, we could choose a nice line in kP^{n-1} as things of the form $\begin{pmatrix} \ast & \ast & 0 \end{pmatrix}^T$. Then, $P_{3,2}$ would be things of the form

$$\begin{pmatrix} \ast & \ast & \ast \\ \ast & \ast & \ast \\ 0 & 0 & \ast \end{pmatrix}\begin{pmatrix} \ast \\ \ast \\ 0 \end{pmatrix} \;=\; \begin{pmatrix} \ast \\ \ast \\ 0 \end{pmatrix}.$$

Thus,

$$P_{3,2} \;=\; \{A \in GL(3) : a_{31}, a_{32} = 0\}.$$

The cases in k^3 aren't very illuminating, aside from showing that $P_{3,1} \cong P_{3,2}$ as a group. However, they are not conjugate, which is to say there is no $g \in GL(3)$ such that

$$gP_{3,1}g^{-1} \;=\; P_{3,2}.$$

Let's briefly look at a slightly larger case.

Example 2. Now we will work in k^4. Through the process shown above, we have

$$\begin{pmatrix} \ast & \ast & \ast & \ast \\ 0 & \ast & \ast & \ast \\ 0 & \ast & \ast & \ast \\ 0 & \ast & \ast & \ast \end{pmatrix} \;\in\; P_{4,1}, \qquad \begin{pmatrix} \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \\ 0 & 0 & \ast & \ast \\ 0 & 0 & \ast & \ast \end{pmatrix} \;\in\; P_{4,2},$$

and finally

$$\begin{pmatrix} \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \\ 0 & 0 & 0 & \ast \end{pmatrix} \in P_{4,3}.$$

Notice that in each case, we replace the lower left corner by a zero matrix with $n - j$ rows and j columns. This leads to the general result.

Theorem. *For a field k,*

$$P_{n,j} = \left\{ \left(\begin{array}{c|c} X & Y \\ \hline 0 & Z \end{array} \right) : X \in GL(j), Z \in GL(n-j) \right\}.$$

This clearly means that $P_{n,j}$ is a subgroup of $GL(n)$.

3. Consequences

If $k = \mathbb{R}$, any linear algebraic group is a manifold, so we can speak of its dimension. If H is a linear algebraic subgroup of G, then G/H is also, a manifold, and

$$\dim(G/H) \;=\; \dim G - \dim H.$$

However, for an arbitrary field k, a linear algebraic group need not be a manifold, but is instead called an (affine) algebraic variety. If $H \subseteq G$ is also a linear algebraic group, then G/H is an algebraic variety as well - but not affine. Like the case for $\mathbb{R}$, however, we can find the dimension as

$$\dim(G/H) \;=\; \dim G - \dim H.$$

From this, we can show that

Theorem. *The dimension of a Grassmannian is given by*

$$\dim\left(Gr(n,j)\right) \;=\; j(n-j).$$

Proof. Recall that elements of $P_{n,j}$ are matrices in GL_n which have an $n-j$ by j zero matrix in the lower left corner, so

$$\dim P_{n,j} \;=\; n^2 - (n-j)j.$$

Utilizing our isomorphism,

$$\begin{aligned}
\dim\left(Gr(n,j)\right) &= \dim\left(GL(n)/P_{n,j}\right) \\
&= \dim GL(n) - \dim P_{n,j} \\
&= n^2 - \big(n^2 - (n-j)j\big) \\
&= (n-j)j.
\end{aligned}$$

□

When we look at dimensions of Grassmannians, there's something akin to Pascal's triangle lurking about. If we construct something of the form

$$\begin{array}{ccccccc}
 & & & \dim(GR(0,0)) & & & \\
 & & \swarrow & & \searrow & & \\
 & \dim(GR(1,0)) & & & & \dim(GR(1,1)) & \\
\swarrow & & \searrow & & \swarrow & & \searrow \\
\dim(GR(2,0)) & & & \dim(GR(2,1)) & & & \dim(GR(2,2)) \\
 & & & \vdots & & &
\end{array}$$

$$\begin{array}{cccc}
\dim(GR(3,0)) & \dim(GR(3,1)) & \dim(GR(3,2)) & \dim(GR(3,3)) \\
\ldots & \ldots & \ldots & \ldots
\end{array}$$

this numerically looks like

$$\begin{array}{ccccccccccccc}
 & & & & & & 0 & & & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & 0 & & 1 & & 0 & & & & \\
 & & & 0 & & 2 & & 2 & & 0 & & & \\
 & & 0 & & 3 & & 4 & & 3 & & 0 & & \\
 & 0 & & 4 & & 6 & & 6 & & 4 & & 0 & \\
0 & & 5 & & 8 & & 9 & & 8 & & 5 & & 0 \\
 & \ddots & & \ddots & & \ddots & & \ddots & & \ddots & & \ddots &
\end{array}$$

which is a rotated version of the multiplication table!

However, a Pascal's triangle also shows up. If $k = \mathbb{F}_q$, where $q = p^n$ for some prime p, we get what is known as a q-deformed Pascal's triangle. Recall the decomposition of projective space into Bruhat cells:

$$kP^n \cong k^n + k^{n-1} + \cdots + k^0.$$

If $k = \mathbb{F}_q$, then

$$\begin{aligned} |kP^n| &= |k^n| + \left|k^{n-1}\right| + \cdots + \left|k^0\right| \\ &= q^n + q^{n-1} + \cdots + 1 \\ &= \frac{q^{n+1} - 1}{q - 1} \\ &= [n+1]_q\,, \end{aligned}$$

the q-integer we defined previously. But we already know that $kP^{n-1} = Gr(n,1)$. Thus $|Gr(n,1)| = [n]_q$. How does this fact generalize?

Definition. The <u>q-factorial</u> $[n]_q!$ is given by

$$[n]_q! = [n]_q \cdot [n-1]_q \cdot \cdots \cdot [1]_q\,,$$

and the <u>q-binomial coefficient</u> is given by

$$\binom{n}{j}_q = \frac{[n]_q!}{[j]_q! \cdot [n-j]_q!}.$$

We will finally state without proof (until the next lecture),

Theorem. *If $k = \mathbb{F}_q$, then*

$$\dim(Gr(n,j)) = \binom{n}{j}_q.$$

Note the interesting analogy: $\binom{n}{j}$ counts the number of j-element subsets of a set of size n, while $\binom{n}{j}_q$ counts the number of j-dimensional subspaces in $\mathbb{F}_q^n$.

In some mysterious sense, a vector space over the "field with one element" (so $q = 1$) is just a finite set!